Gradient Methods and Conic Least Squares Problems

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Abstract

This paper presents two reformulations of the dual of the constrained least squares problem over convex cones. In addition, it extends Nesterov’s excessive gap method 1 [21] to more general problems. The conic least squares problem is then solved by applying the resulting modified method, or Nesterov’s smooth method 2 [22], or Nesterov’s excessive gap method 2 [21], to the dual reformulations. Numerical experiments show that this approach obtains relatively accurate solutions for large-scale problems using less CPU time than interior-point method based state-of-art software do and is more accurate than these software on problem instances possibly with dual degeneracy [2]. The approach extends easily to related conic convex quadratic programs.

Keywords: Conic least squares problem, Semidefinite least squares problem, Conic convex quadratic program, Nesterov’s smoothing algorithm, Nesterov’s excessive gap technique, Fast gradient methods, Lagrange dual.

Mathematics Subject Classification (MSC2010): 90C25, 90C20, 90C22, 90C51, 90C06.

1 Introduction

Nesterov in [21, 22] developed smooth approximation and excessive gap technique for some structured non-smooth optimization problems, ended up with gradient schemes having efficiency estimates $O(1/\varepsilon)$, where $\varepsilon$ is the desired accuracy of the solution. This complexity is one magnitude faster than the theoretical lower complexity bound $O(1/\varepsilon^2)$ of the subgradient method for the black-box oracle model of non-smooth convex minimization.

Let $Q_1$ and $Q_2$ be closed convex sets over finite-dimensional normed real-vector spaces $E_1$ and $E_2$ respectively. The dual space of $E_2$ is denoted as $E_2^*$. Below is the structured nonsmooth optimization problem considered in [21, 22]:

\begin{equation}
\text{Find } f^* = \min_{x \in Q_1} f(x), \quad f(x) = \hat{f}(x) + \max_{u \in Q_2} \{ \langle Hx, u \rangle - \hat{\psi}(u) \},
\end{equation}

where the function $\hat{f}(x)$ is convex on $Q_1$ and has a Lipschitz continuous gradient with Lipschitz constant $L(\hat{f})$, and $\hat{\psi}(u)$ is a continuous convex function on $Q_2$. The linear operator $H$ maps $E_1$ to $E_2^*$. The adjoint form of (1.1) is

\begin{equation}
\max_{u \in Q_2} \{ \psi(u) \}, \quad \psi(u) = -\hat{\psi}(u) + \min_{x \in Q_1} \{ \langle Hx, u \rangle + \hat{f}(x) \}.
\end{equation}

Let $\phi(u)$ be a prox-function of the feasible set $Q_2$, i.e. $\phi(u)$ is continuous and strongly convex on $Q_2$ with convexity parameter $\sigma > 0$. Let $\bar{u} = \arg \min_{u \in Q_2} \phi(u)$ be the center of $Q_2$. Assume, without loss of generality, that $\phi(\bar{u}) = 0$. Thus, for any $u \in Q_2$ we have

$\phi(u) \geq \frac{\sigma}{2} \| u - \bar{u} \|^2.$

To solve (1.1), in [22], Nesterov approximates $f(x)$ by a smooth function with a Lipschitz continuous gradient:

\begin{equation}
f_\mu(x) \equiv \hat{f}(x) + \max_{u \in Q_2} \{ \langle Hx, u \rangle - \psi(u) - \mu \phi(u) \}, \quad x \in Q_1.
\end{equation}

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He then minimizes the smooth function by a fast gradient scheme having an efficiency estimate of the order $O(\sqrt{\epsilon})$, where $L$ is the Lipschitz constant for the gradient of the objective function $f_\mu$. The $L$ in the smooth $\epsilon$-approximation of the initial function can be chosen of the order $O(1/\epsilon)$. Thus, Nesterov gives a gradient scheme for (1.2) with an efficiency estimate of the order $O(1/\epsilon)$.

Let $L_2(\bar{\psi})$ be the Lipschitz constant of $\bar{\psi}$. The excessive gap technique [21] is designed for two special cases of the primal-dual pair (1.1)–(1.2): METHOD 1 for problem instances with both $L_1(\hat{f})$ and $L_2(\bar{\psi})$ being 0, and METHOD 2 for problem instances with $\hat{f}(x)$ being strongly convex. Let $\phi_1(x)$ be a prox-function of the feasible set $Q_1$ with convexity parameter $\sigma_1 > 0$. Let $\bar{x} = \arg\min_{x \in Q_1} \phi_1(x)$ be the center of $Q_1$. For any $\mu \geq 0$, denote

$$
\psi_\mu(u) = -\bar{\psi}(u) + \min_{x \in Q_1} \left\{ \langle Hx, u \rangle + \bar{f}(x) + \mu \phi_1(x) \right\}, \quad u \in Q_2.
$$

In the excessive gap methods [21], for each iterate $(\bar{x}_k, \bar{u}_k)$, Nesterov maintains the excessive gap condition:

$$
f_{\mu_{k+1}}(\bar{x}_k) \leq \psi_{\mu_k}(\bar{u}_k), \quad k = 0, 1, \ldots
$$

Let $u(x)$ be a solution of $f(x)$ in $u$ and $x(u)$ be a solution of $\psi(u)$ in $x$:

$$
u(x) \in \arg\max_{u \in Q_2} \left\{ \langle Hx, u \rangle - \bar{\psi}(u) \right\}, \quad x(u) \in \arg\min_{x \in Q_1} \left\{ \langle Hx, u \rangle + \bar{f}(x) \right\}.
$$

For any pair $(\bar{x}, \bar{u})$ satisfying the excessive gap condition (1.5) with smoothness parameters $\mu_1$ and $\mu_2$, from definitions (1.4) and (1.6), we have

$$
\psi_{\mu_k}(\bar{u}) = -\bar{\psi}(\bar{u}) + \min_{x \in Q_1} \left\{ \langle Hx, \bar{u} \rangle + \bar{f}(x) + \mu_1 \phi_1(x) \right\} \leq -\bar{\psi}(\bar{u}) + \left\{ \langle Hx(\bar{u}), \bar{u} \rangle + \bar{f}(x(\bar{u})) + \mu_1 \phi_1(x(\bar{u})) \right\} \leq \psi(\bar{u}) + \mu_1 \phi_1(x(\bar{u})).
$$

Therefore,

$$
f(\bar{x}) - \mu_2 \phi[u(\bar{x})] \leq f_{\mu_2}(\bar{x}) \leq \psi(\bar{u}) + \mu_1 \phi_1(x(\bar{u})).
$$

Hence

$$
f(\bar{x}) - \psi(\bar{u}) \leq \mu_1 \phi_1(x(\bar{u})) + \mu_2 \phi[u(\bar{x})].
$$

Further in view of weak duality $\psi(u) \leq f(x)$ (for any $x \in Q_1$ and $u \in Q_2$), we obtain (cf. [21, Lemma 3.1])

$$
0 \leq f(\bar{x}) - \psi(\bar{u}) \leq \mu_1 \phi_1(x(\bar{u})) + \mu_2 \phi[u(\bar{x})].
$$

Assume that $\phi_1(x(u))$ and $\phi[u(x)]$ are bounded for any $x \in Q_1$ and $u \in Q_2$. In the excessive gap methods [21], Nesterov drives $\mu_{k+1}$ and $\mu_{2k+1}$ to 0 by a switching strategy to obtain approximate solutions for $\min_{u \in Q_2} f(x)$ and $\max_{u \in Q_2} \psi(u)$. The complexity estimate of the Excessive Gap METHOD 1 is of order $O(1/\epsilon)$, and that of the Excessive Gap METHOD 2 is of order $O(1/\sqrt{\epsilon})$.

This paper extends the Excessive Gap METHOD 1 [21] to problem (1.1)–(1.2) where $L_1(\hat{f})$ and $L_2(\bar{\psi})$ are nonnegative instead of being restricted to zero as in [21], and the complexity estimate of the modified scheme remains of the same order—$O(1/\epsilon)$—as that in [21].

The modified Excessive Gap METHOD 1 as well as Nesterov’s smooth method [22] and Excessive Gap METHOD 2 are then applied to the following conic least squares problem: Let $x$ stands for the column vector composed of column vectors $x_1, \ldots, x_n$. Denote linear operators $A = [A_1 \ldots A_m]$, $E = [E_1 \ldots E_m]$, $G = [G_1 \ldots G_m]$ with $A_i$, $E_i$ and $G_i$ corresponding to $x_i$. Let $Idx$ be a subset of the index set $\{1, \ldots, n\}$. For each $i \in Idx$, $\mathcal{K}_i$ is a closed convex cone, and $d_i$ is a constant vector of the same dimension as that of $x_i$. Let $\mathcal{K}$ represent the Cartesian product of $\mathcal{K}_i$: $\mathcal{K} = \bigotimes_{i \in Idx} \mathcal{K}_i$, which is also a cone. The conic least squares problem is:

$$
\begin{aligned}
\min_{x} & \quad \|Ax - b\|^2 \\
\text{s.t.} & \quad Ex = f \\
& \quad Gx \leq h \\
& \quad x_i - d_i \in \mathcal{K}_i, \quad i \in Idx,
\end{aligned}
$$

where the number of equalities is $m$, the number of inequalities is $q$, and the number of rows of $A$ is $p$. Let $N$ be the total dimension of variable $x$, and let $N_d$ represent the sum of dimensions of all variables $x_i$ with index $i \in Idx$. 

\[ \text{dim}(x) = \sum_{i \in Idx} \text{dim}(x_i). \]
Problem (1.8) includes several popular models: For $\mathcal{K}$ being the non-negative orthant, the model is the classical non-negativity constrained least squares problem [17]; for $\text{Id}x = 0$, the model is the inequality constrained least-squares estimation discussed in [17], where prior and sample information is combined in a regression equation. The semidefinite least squares (SDLS) problem studied in [13, 19, 33], among others, is a special case of model (1.8), where $\mathcal{K}$ is the cone of positive semidefinite symmetric matrices, the operator $A$ is the identity, and the set $\text{Id}x$ is $\{1, \ldots, n\}$; see [11] for theory, algorithms and applications of SDLS.

In many applications the operator $A$ in model (1.8) is a general linear operator—not restricted to the identity $I$ as in the SDLS model. These include the computation of the nearest correlation matrix in H-norm (Hadamard product norm) [13], the weighted H-norm Euclidean matrix completion problem [1], finding the asymptotic distribution of the test statistic on assessing the goodness of fit of moment structural models, Markowitz portfolio-selection [20], isotonic regression [3], the estimation of an aggregate demand function in economy [14], etc.

Numerical iterative methods are usually employed to find an approximate solution of problem (1.8). For $\mathcal{K}$ being the cone of positive semidefinite symmetric matrices, the least squares problem (1.8) can be transformed into a standard linear-quadratic-semidefinite program or a semidefinite linear complementarity problem, both of which are approximately solvable by interior-point methods [8, 16, 23, 24, 30, 32]; but as is already analyzed by several researchers [13, 19, 30], because of the large number of additional variables, the dimension of the reformulations increases dramatically, which causes difficulty to current software. For large-scale problems, just fitting the Newton system from each iteration of an interior-point method into a computer memory is impractical. In addition, even if the operator $A$ has a nice structure, the Newton system at each iteration is likely difficult to solve, because the Newton system is formed by operator $A$ coupled with equality constraints. Furthermore, to have a numerically convergent solution, the problem instance must satisfy primal-dual non-degeneracy and strict complementarity conditions [2]. The focus of paper [30] is to design efficient preconditioners for iterative Krylov subspace solvers for the Newton system at each iteration of an interior-point method for the convex quadratic semidefinite program with a non-singular quadratic term (corresponding to $A$ having a full column rank) under primal-dual nondegeneracy and strict complementarity assumptions. That paper [30] notes that the computational complexity of each iteration of an interior-point method is inherently higher than that of a first-order method, and the convergence of an iterative Krylov subspace method depends on the condition number of the linear system at each iteration, which in turn depends on problem instances, and there is no good estimation of the convergence rate of Krylov subspace iteration methods.

Some non-interior point methods ([9, 19, 27, 28] etc.) have been proposed to solve the SDLS problem. These algorithms have their own advantages, but since they are based on the dual studied in [19], it is observed in [30] that because there is no obvious way of extending that dual form to problem (1.8) having a general operator $A$, their approaches cannot be used to solve general cases of problem (1.8). The approach in [28] is only applicable to cases satisfying two additional technical conditions and with the $\mathcal{K}$ being self-dual; although the semismooth Newton’s method has locally Q-quadratic convergence—good for warm-starting [34]. The computational complexity of each iteration of a semismooth Newton’s method is about the same order as that of an interior-point method. Furthermore, it is cumbersome to deal with inequality constraints by Newton’s methods. In [6], the dual presented in [19] with additional inequality constraints is solved by a gradient projection method. But it only works on cases where the operator $A = I$. The alternating projection method has difficulty dealing with a general operator $A$, too; see [13].

This paper derives the Lagrange dual of problem (1.8) and gives some reformulations of the dual. The primal problem (1.8) is solved by applying the aforementioned Nesterov’s smooth method, modified Nesterov’s excessive gap METHOD 1, and Nesterov’s excessive gap METHOD 2 to the reformulations. This approach is more efficient than directly solving the original problem by the gradient methods on the numerical example given in this paper. For problem instances of model (1.8) satisfying the Primal Range-Space Condition—the operator $A$ having full column rank is a special case, the proposed scheme has an efficiency estimate $O(1/\sqrt{\varepsilon})$ with Nesterov’s smooth method or excessive gap METHOD 2; otherwise, the scheme has an efficiency estimate $O(1/\varepsilon)$ with modified Nesterov’s excessive gap METHOD 1, in which case Nesterov’s original excessive gap technique is inapplicable. The approach given in this paper only uses the function values and the gradients of the objective function; so the complexity of each iteration for this approach is lower than that of a second-order method. In addition, each iteration is decomposable into smaller problems which can be solved in parallel. On the other hand, these fast gradient methods converge faster than the standard gradient methods. The model (1.8) is more general than those in other papers ([6, 9, 13, 19, 27, 28, 30, 33] etc.): The cone $\mathcal{K}$ is only assumed to be closed and convex and that it is easy to compute the projection onto the dual cone $\mathcal{K}^*—\mathcal{K}$ needs not to be self dual; the operator $A$ can be any linear matrix—not necessarily having a full-column rank; the problem can have many equality and inequality constraints and free variables; neither strict complementarity nor primal-dual non-degeneracy is assumed for the problem. Numerical results show that using less than 3% of the total CPU time needed by SDPT3 [31], the approach proposed in the paper
can obtain relatively accurate solutions on large-scale problems which Sedumi [29] can’t handle; and the approach is more accurate than these state-of-art software on problem instances possibly lacking primal-dual nondegeneracy or strict complementarity. With the increase in the numbers of equality and inequality constraints, the gradient methods are more and more advantageous than the second-order methods. The approach extends easily to related conic convex quadratic programs.

The remainder of the paper is organized as follows. In §2, the Lagrange dual of model (1.8) is studied, and some reformulations of the dual are provided. In §3, Nesterov’s smooth method [22] is applied to the reformulations. In §4, Nesterov’s excessive gap technique METHOD 1 [21] is extended to more general problems. Then the modified method and Nesterov’s excessive gap technique METHOD 2 are applied to the reformulations. In §5, some numerical results are presented.

2 The Reformation

This section reformulates model (1.8) based on the Lagrange multiplier theory and the generalized inverse of a matrix so that the reformulations have the structure of model (1.1). The relevant properties of the Moore-Penrose generalized inverse is provided in Appendix A. Two reformulations are presented in §§2.2 and §§2.3. Then this approach is extended to conic convex quadratic programs in §§2.4.

Some notations. Let \( 	ilde{x} \) be a sub-vector of \( x \) including all \( x_i \) with \( i \in Idx \). Without loss of generality, assume that \( \tilde{x} \) is the last part of \( x \). For a matrix \( M \) and an index set \( S \), let \( \{ M \}_S \) represent the submatrix of \( M \) having row indices from \( S \).

Let \( \mathcal{K}^* \) denote the dual cone of \( \mathcal{K} \):

\[
\mathcal{K}^* = \{ y : \langle y, x \rangle \geq 0, \forall x \in \mathcal{K} \}.
\]

Because \( \mathcal{K} \) is closed and convex, it follows that \( (\mathcal{K}^*)^* = \mathcal{K} \).

Denote the non-negative orthant: \( \mathbb{R}_+ \equiv \{ t \in \mathbb{R} : t \geq 0 \} \).

For a linear operator \( M \), let \( M^* \) denote its adjoint, let \( \mathcal{K}(M) \) denote the range space of \( M \), let \( M^2 \) represent the null space of \( M \), let \( M^\dagger \) represent its Moore-Penrose generalized inverse (Appendix A), let \( \| M \|_2 \) represent the operator norm induced by the vector norm: \( \| M \|_2 = \max_{\| x \| = 1} \| Mx \| = \lambda_{\text{max}}(M^*M)^{1/2} \).

2.1 The Lagrangian Dual

The Lagrange function of the least squares problem (1.8) is

\[
L(x, y, \tilde{v}, \tilde{z}) = \| Ax - b \|^2 + \langle y, Ex - f \rangle + \langle v, Gx - h \rangle - \langle \tilde{z}, \tilde{x} \rangle - \langle \tilde{d}, \tilde{d} \rangle, \quad (v \geq 0, \tilde{z} \in \mathcal{K}^*).
\]

If problem (1.8) satisfies some constraint qualification, such as one of those studied in [5, 10, 12, 15, 35], a finite Lagrange multiplier exists, and we have

\[
\min_x \max_{(y, v, \tilde{z})} L(x, y, v, \tilde{z}) = \max_{(y, v, \tilde{z})} \min_x L(x, y, v, \tilde{z}) = -\min_{(y, v, \tilde{z})} \left\{ \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{d} \rangle + \max_x \left[ -\| Ax - b \|^2 - \langle E, y \rangle - \langle G, v \rangle + \langle \tilde{z}, \tilde{x} \rangle \right] \right\}.
\]

Thus, solving problem (1.8) is equivalent to solving

\[
(2.1) \min_{(y, v, \tilde{z})} \left\{ \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{d} \rangle + \max_x \left[ -\| Ax - b \|^2 - \langle E, y \rangle - \langle G, v \rangle + \langle \tilde{z}, \tilde{x} \rangle \right] \right\}.
\]

Writing down the first-order optimality condition in \( x \) for the saddle-point problem (2.1), we obtain

\[
(2.2a) \min_{u, y, \tilde{v}, \tilde{z}} \langle Au, u \rangle + \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle - \| b \|^2
\]

\[
(2.2b) \quad \text{s.t.} \quad 2A^*u + E^*y + G^*v - \begin{pmatrix} 0 \\ \tilde{z} \end{pmatrix} = 2A^*b
\]

\[
(2.2c) \quad \tilde{z} \in \mathcal{K}^*, \quad v \geq 0.
\]

To design efficient algorithms for problem (1.8), next we give two reformulations of model (2.2).
2.2 The Primal Range-Space Condition

When $A$ does not have full column rank, the solution in $x$ for the Lagrange (2.1) is not unique. By [26], given a triple $(\tilde{z}, v, y)$, the minimum-norm solution in $x$ for (2.2) is

$$x(\tilde{z}, v, y) = u = A^\dagger b - (A^*A)^\dagger \left[ \frac{1}{2} E^* y + \frac{1}{2} G^* v - \frac{1}{2} \left( \begin{array}{c} 0 \\ \tilde{z} \end{array} \right) \right].$$

The space $\mathbb{R}^N$ is a direct sum of $\mathcal{R}(A^*)$ and the null space of $A$. The projection onto $\mathcal{R}(A^*)$, denoted as $P_{A^*}$, is

$$P_{A^*} \overset{(A.6)}{=} A^* A^+ (\overset{(A.7)}{=} A^\dagger A \overset{(A.3)}{=} A^\dagger A^+ A^* A \overset{(A.4)}{=} (A^* A)^\dagger (A^* A).$$

And the projection onto $A^\perp$ is

$$P_{A^\perp} = I - A^\dagger A = I - (A^* A)^\dagger (A^* A).$$

Applying $P_{A^\perp}$ to both sides of equation (2.2b), and in view of property (A.3), we have

$$2 A^* A u + E^* y + G^* v - \left( \begin{array}{c} 0 \\ \tilde{z} \end{array} \right) = 2 A^* b, \quad (I - A^\dagger A) (E^* y + G^* v - \left[ \begin{array}{c} 0 \\ \tilde{z} \end{array} \right]) = 0.$$

From the definition of Moore-Penrose generalized inverse (A.1), it follows that $(A^* A)^+ (A^* A)^\dagger (A^* A) = A^* A$. Therefore,

$$(Au, Au) = \langle A^* Au, (A^* A)^\dagger (A^* A) u \rangle.$$

Substituting the first equality in (2.4) into objective (2.2a) and dropping $\|b\|^2$, we can represent problem (2.2) as

$$\begin{align*}
\min_{\tilde{z}, v, y} & \quad F(\tilde{z}, v, y) = \langle A^* b + \frac{1}{2} \left( \begin{array}{c} 0 \\ \tilde{z} \end{array} \right) - \frac{1}{2} G^* v - \frac{1}{2} E^* y, (A^* A)^\dagger \left[ A^* b + \frac{1}{2} \left( \begin{array}{c} 0 \\ \tilde{z} \end{array} \right) - \frac{1}{2} G^* v - \frac{1}{2} E^* y \right] \rangle \\
& \quad + \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle \\
\text{s.t.} & \quad (I - A^\dagger A) (E^* y + G^* v - \left[ \begin{array}{c} 0 \\ \tilde{z} \end{array} \right]) = 0, \\
& \quad v \geq 0, \tilde{z} \in \mathcal{K}^*.
\end{align*}$$

Define the Primal Range-Space Condition as follows:

$$(I - A^\dagger A) (E^* y + G^* v - \left[ \begin{array}{c} 0 \\ \tilde{z} \end{array} \right]) = 0 \quad \text{for any } y \in \mathbb{R}^m, v \geq 0, \tilde{z} \in \mathcal{K}^*.$$

The Primal Range-Space Condition is satisfied, for instance, when each column of $\left[ \begin{array}{c} E^* \\ G^* \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \end{array} \right]$ is in $\mathcal{R}(A^*)$, which is implied if $A$ has full column rank.

**Proposition 2.1.** The minimization of the Lagrange function with respect to $x$: $\min_x L(x, y, v, \tilde{z})$ is attained for any fixed $y \in \mathbb{R}^m, v \geq 0, \tilde{z} \in \mathcal{K}^*$, iff the Primal Range-Space Condition is satisfied.

**Proof.** We have

$$\begin{align*}
L(x, y, v, \tilde{z}) &= - \langle f, y \rangle - \langle h, v \rangle + \langle \tilde{d}, \tilde{z} \rangle + \|Ax - b\|^2 + \langle Ex, y \rangle + \langle Gx, v \rangle - \langle \tilde{x}, \tilde{z} \rangle \\
&= \|Ax - b\|^2 + \langle Ax, (EA^*)^* y \rangle + \langle Ax, (GA^*)^* v \rangle - \langle Ax, \left[ \begin{array}{c} 0 \\ I \end{array} \right] A^\dagger \rangle \tilde{z}^* + \langle (I - A^\dagger A)^* E^* y, x \rangle \\
&\quad + \langle (I - A^\dagger A)^* G^* v, x \rangle - \langle (I - A^\dagger A)^* \left[ \begin{array}{c} 0 \\ \tilde{z} \end{array} \right], x \rangle - \langle f, y \rangle - \langle h, v \rangle + \langle \tilde{d}, \tilde{z} \rangle \\
&\overset{(A.2)}{=} \|Ax - b + \frac{1}{2} (EA^*)^* y + \frac{1}{2} (GA^*)^* v - \frac{1}{2} \left( \begin{array}{c} 0 \\ I \end{array} \right) A^\dagger \rangle \tilde{z}^*\|^2 + \langle (I - A^\dagger A)^* (E^* y + G^* v - \left[ \begin{array}{c} 0 \\ \tilde{z} \end{array} \right], x \rangle \\
&\quad - \frac{1}{4} \| (EA^*)^* y + (GA^*)^* v - \left[ \begin{array}{c} 0 \\ I \end{array} \right] A^\dagger \rangle \tilde{z}^*\|^2 + \langle b, (EA^*)^* y \rangle + \langle b, (GA^*)^* v \rangle - \langle b, A^\dagger \left[ \begin{array}{c} 0 \\ \tilde{z} \end{array} \right] \rangle \\
&\quad - \langle f, y \rangle - \langle h, v \rangle + \langle \tilde{d}, \tilde{z} \rangle.
\end{align*}$$

If there exist $\tilde{y} \in \mathbb{R}^m, \tilde{v} \geq 0, \tilde{z} \in \mathcal{K}^*$ such that

$$w \equiv (I - A^\dagger A) (E^* \tilde{y} + G^* \tilde{v} - \left[ \begin{array}{c} 0 \\ \tilde{z} \end{array} \right]) \neq 0,$$
then letting $\tilde{x} = -\alpha w$, by the definition of Moore-Penrose generalized inverse (A.1) in Appendix A, we have $A\tilde{x} = 0$.

Therefore,

$$L(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{z}) = \| -b + \frac{1}{2}(EA^\dagger)^*\tilde{y} + \frac{1}{2}(GA^\dagger)^*\tilde{v} - \frac{1}{2} \left( \begin{bmatrix} 0 & 1 \end{bmatrix} A^\dagger \right)^* \tilde{z} \| - \alpha u^2 - \frac{1}{4} \| (EA^\dagger)^*\tilde{y} + (GA^\dagger)^*\tilde{v} - \left( \begin{bmatrix} 0 & 1 \end{bmatrix} A^\dagger \right)^* \tilde{z} \|^2$$

$$+ \langle b, (EA^\dagger)^*\tilde{y} \rangle + \langle b, (GA^\dagger)^*\tilde{v} \rangle - \langle b, A^\dagger \begin{bmatrix} 0 \\ z \end{bmatrix} \tilde{z} \rangle - \langle f, \tilde{y} \rangle - \langle h, \tilde{v} \rangle + \langle \tilde{d}, \tilde{z} \rangle.$$ 

Thus, $\lim_{\alpha \to +\infty} L(\tilde{x}, \tilde{y}, \tilde{v}, \tilde{z}) = -\infty$.

On the other hand, for any $\tilde{y} \in \mathbb{R}^m, \tilde{v} \geq 0, \tilde{z} \in \mathcal{K}^*$ such that $(I - A^\dagger A) (E^*\tilde{y} + G^*\tilde{v} - \left[ \begin{bmatrix} 0 \\ z \end{bmatrix} \tilde{z} \right] ) = 0$, then

$$L(x, \tilde{y}, \tilde{v}, \tilde{z}) = \| Ax - b + \frac{1}{2}(EA^\dagger)^*\tilde{y} + \frac{1}{2}(GA^\dagger)^*\tilde{v} - \frac{1}{2} \left( \begin{bmatrix} 0 & 1 \end{bmatrix} A^\dagger \right)^* \tilde{z} \| - \alpha u^2 - \frac{1}{4} \| (EA^\dagger)^*\tilde{y} + (GA^\dagger)^*\tilde{v} - \left( \begin{bmatrix} 0 & 1 \end{bmatrix} A^\dagger \right)^* \tilde{z} \|^2$$

$$+ \langle b, (EA^\dagger)^*\tilde{y} \rangle + \langle b, (GA^\dagger)^*\tilde{v} \rangle - \langle b, A^\dagger [\begin{bmatrix} 0 \\ z \end{bmatrix} \tilde{z}] \rangle - \langle f, \tilde{y} \rangle - \langle h, \tilde{v} \rangle + \langle \tilde{d}, \tilde{z} \rangle,$$

which is a closed convex quadratic function in $x$; so it attains its minimum of a finite value.

\begin{proposition}
Under the Primal Range-Space Condition, problem (2.5) can be represented as

(2.6) \[ \min_{z \in \mathcal{K}^*, v \geq 0, y \in \mathbb{R}^m} F(z, v, y). \]

\end{proposition}

\begin{lemma}
A necessary and sufficient condition for problem (2.6) having a bounded optimal value and a non-empty solution set is $f \in \mathcal{R}(EA^\dagger), h \in \mathcal{R}(GA^\dagger) + \mathbb{R}_+, \tilde{d} \in \mathcal{R}(A^\dagger I_{|d|}) - \mathcal{K}$, and the set $\{ u \in \mathbb{R}^p : h - GA^\dagger u \geq 0, f - EA^\dagger u = 0, \{ A^\dagger \}_{I_{|d|}u - \tilde{d} \in \mathcal{K}} \} \text{ is non-empty.} $

\begin{proof}
By equation (A.4) in Appendix A, we have

$$\min_{z \in \mathcal{K}^*, v \geq 0, y \in \mathbb{R}^m} F(z, v, y) = \min_{y \in \mathbb{R}^m, v \geq 0, z \in \mathcal{K}^*} \max_{u \in \mathbb{R}^p} 2 \langle A^\dagger b + \frac{1}{2} \begin{bmatrix} 0 \\ z \end{bmatrix} \rangle - \frac{1}{2} G^* v - \frac{1}{2} E^* y \rangle u \| \| - u \|^2$$

$$+ \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle$$

$$= \max_{u \in \mathbb{R}^p} \min_{y \in \mathbb{R}^m, v \geq 0, z \in \mathcal{K}^*} 2 \langle u, A^\dagger A^\dagger u \rangle - \| u \|^2 + \langle h - GA^\dagger u, v \rangle + \langle f - EA^\dagger u, y \rangle$$

$$+ \langle \tilde{d}, \tilde{z} \rangle.$$ 

For the above last expression being bounded, the inner min must have a finite solution, which is satisfied iff $h - GA^\dagger u \geq 0, f - EA^\dagger u = 0, \{ A^\dagger \}_{I_{|d|}u - \tilde{d} \in \mathcal{K}} \text{ is non-empty.} $ \hfill \qed
\end{proof}

\subsection{The Succinct Dual Form}
Since $y$ is unconstrained in problem (2.6), we can simplify formulation (2.6) by eliminating $y$ from it.

\begin{lemma}
Problem (2.6) has a unique solution in $y$ iff $EA^\dagger$ has rank $m$. Suppose $f \in \mathcal{R}(EA^\dagger)$. Then a minimum-norm solution in $y$ for problem (2.6) is

(2.7) \[ y = (EA^\dagger)^\dagger \left\{ 2A^\dagger b + \begin{bmatrix} 0 \\ z \end{bmatrix} \right\} - 2(EA^\dagger)^\dagger f \]. \]

\end{lemma}

\begin{proof}
For fixed $(\tilde{z}, v)$, to emphasize that the variable is $y$, let us denote the objective function of (2.6) with respect to $y$ as $F_{\tilde{z}, v}(y)$.

Since

$$\langle f, y \rangle = \langle (EA^\dagger)^\dagger f, (EA^\dagger)^\dagger y \rangle + \langle \left\{ I - (EA^\dagger)(EA^\dagger)^\dagger \right\} f, y \rangle,$$
we can write $F_{\tilde{z},v}(y)$ as

$$F_{\tilde{z},v}(y) = \frac{1}{2} \left\| (EA)^* y - A_t x \left[ A^* b + \frac{1}{2} \left( \frac{0}{\tilde{z}} \right) - \frac{1}{2} G^* v \right] \right\|^2 + \left( (EA)^t f, (EA)^t y \right)$$

$$+ \left( \left[ I - (EA)^t(EE)^t \right] f, y \right) + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle$$

(2.8)

$$= \frac{1}{2} \left\| (EA)^* y - A_t x \left[ A^* b + \frac{1}{2} \left( \frac{0}{\tilde{z}} \right) - \frac{1}{2} G^* v \right] + (EA)^t f \right\|^2$$

$$+ \left( \left[ I - (EA)^t(EE)^t \right] f, y \right) + 2 \left( A^t(EE)^t f, A^* b + \frac{1}{2} \left( \frac{0}{\tilde{z}} \right) - \frac{1}{2} G^* v \right)$$

$$- \| (EA)^t f \|^2 + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle.$$

For a scalar $\alpha < 0$, define

$$\tilde{y}(\alpha) \equiv (EA)^t \left\{ A_t x \left[ 2A^* b + \left( \frac{0}{\tilde{z}} \right) - G^* v \right] - 2(EE)^t f \right\} + \alpha \left[ I - (EA)^t(EE)^t \right] f.$$

Since $(EA)^t(EE)^t \equiv 0$ and $(EA)^t \left[ I - (EA)^t(EE)^t \right] \equiv 0,$

we have

$$\frac{1}{2}(EA)^* \tilde{y}(\alpha) = (EA)^t(EE)^t \left\{ A_t x \left[ A^* b + \frac{1}{2} \left( \frac{0}{\tilde{z}} \right) - \frac{1}{2} G^* v \right] - (EA)^t f \right\}$$

and

$$\left( \left[ I - (EA)^t(EE)^t \right] f, \tilde{y}(\alpha) \right) \equiv \alpha \left\| I - (EA)^t(EE)^t \right\| f \|^2.$$

Therefore,

$$F_{\tilde{z},v}[\tilde{y}(\alpha)] = \left\| \left[ I - (EA)^t(EE)^t \right] f, \tilde{y}(\alpha) \right\| \equiv \alpha \left\| I - (EA)^t(EE)^t \right\| f \|^2$$

(2.9)

$$+ \left( \left[ I - (EA)^t(EE)^t \right] f, \tilde{y}(\alpha) \right) \equiv \alpha \left\| I - (EA)^t(EE)^t \right\| f \|^2 + 2 \left( A^t(EE)^t f, A^* b + \frac{1}{2} \left( \frac{0}{\tilde{z}} \right) - \frac{1}{2} G^* v \right)$$

$$- \| (EA)^t f \|^2 + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle.$$

Since $\left[ I - (EA)^t(EE)^t \right]$ is the projection onto the orthogonal space of $\mathcal{R}(EE)$, we have $\left\| I - (EA)^t(EE)^t \right\| f \| = 0$ iff $f \in \mathcal{R}(EA)$. Therefore,

$$\lim_{\alpha \to -\infty} F_{\tilde{z},v}[\tilde{y}(\alpha)] \to -\infty, \quad f \notin \mathcal{R}(EA).$$

On the other hand, when $f \in \mathcal{R}(EA)$, we have $\left[ I - (EA)^t(EE)^t \right] f = 0$; thus by (2.8), the solution to $\min_y F_{\tilde{z},v}(y)$ is a least squares solution to

$$\frac{1}{2} (EA)^* y = A_t x \left[ A^* b + \frac{1}{2} \left( \frac{0}{\tilde{z}} \right) - \frac{1}{2} G^* v \right] - (EA)^t f.$$

By [17], the solution is unique iff $EA^t$ has full row rank—rank $m$. Further by [26], expression (2.7) is a minimum-norm solution of $y$ to the above least squares problem.

For problems with special structures, the dual (2.6) can be further simplified based on expression (2.7).

**Theorem 2.1.** Suppose that problem (1.8) satisfies some constraint qualification and the Primal Range-Space condition. Let $f \in \mathcal{R}(EA)$. Also assume $\tilde{d} \in \mathcal{K} \left( A^t \right)_{ldx} \left[ I - (EA)^t(EE)^t \right] f - \mathcal{K}, h \in \mathcal{K} \left( GA^t(IE^t)(EA)^t \right) + GA^t(EE)^t f + \mathbb{R}^m_+$, and the set $u \in \mathbb{R}^p$ : $\left( A^t \right)_{ldx} \left[ I - (EA)^t(EE)^t \right](u + b) + \left( A^t \right)_{ldx}(EA)^t f - \tilde{d} \in \mathcal{K}$. 

---

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By (A.2) and (A.4) in Appendix A, we obtain the reformulation (2.10).

\[
\min_{z \in \mathcal{K}^*, v \geq 0} \left\{ F_i(z, v) = \frac{1}{4} \left( \langle 0, z \rangle - G^* v \right) \right. \\
\left. + \left( A^* b + A^* (E A^*)^f (f - E A^* b) + \langle h, v \rangle - \langle d, z \rangle \right) \right\}.
\]

(2.10)

In addition, the optimal value of the dual (2.10) is bounded.

**Proof.** Set $y$ by equation (2.7) in the Lagrange dual (2.6). Further in view of equations (2.9) and (A.1), we can simplify the Lagrange dual (2.6) as:

\[
\min_{z \in \mathcal{K}^*, v \geq 0} \left\{ f, (E A^*)^t \left[ \langle 0, z \rangle - G^* v \rangle \right] + \langle h, v \rangle - \langle d, z \rangle + A^* b + \frac{1}{2} \langle 0, z \rangle \right\} \\
- \frac{1}{2} G^* v \left[ A^* (E A^*)^t (E A^*)^t \right] A^* v + \langle h, v \rangle - \langle d, z \rangle.
\]

(2.11)

By equations (A.2) and (A.3), we can get that (2.10) is equivalent to

\[
\min_{z \in \mathcal{K}^*, v \geq 0} \max_{u \in \mathbb{R}^p} \left\{ P \left[ \langle 0, z \rangle - G^* v \rangle \right] , u \right\} - \| u \|^2 + \left\{ P^* b + A^* (E A^*)^t f , \langle 0, z \rangle - G^* v \rangle \right\} + \langle h, v \rangle - \langle d, z \rangle
\]

\[
= \max_{u \in \mathbb{R}^p} \min_{z \in \mathcal{K}^*, v \geq 0} - \| u \|^2 + \left\{ P^* I_{d_x} (u + b) + A^* \right\} I_{d_x} (E A^*)^t f - d, z \rangle + \langle h - G P^* (u + b) - G A^* (E A^*)^t f , v \rangle.
\]

A necessary and sufficient condition for the above inner min problem having a finite optimal value is that the set \{ $u \in \mathbb{R}^p$ : \{ $P^* I_{d_x} (u + b) + A^* \}$ $I_{d_x} (E A^*)^t f - d \in \mathcal{K}, h - G P^* (u + b) - G A^* (E A^*)^t f \geq 0$ \} is non-empty.

Substituting $y$ in equation (2.3) by equation (2.7), we obtain the primal solution of the succinct dual:

\[
x_i(z, v) = A^* b - (A^* A^*)^t E (E A^*)^t f [A^* A^* b - (E A^*)^t f] + \frac{1}{2} (A^* A^*)^t [I - E^* (E A^*)^t A^*] \left[ \langle 0, z \rangle - G^* v \right] 
\]

(2.12)

**Remark.** Let $\mathcal{K}^*$ denote the polar cone of $\mathcal{K}$; $\mathcal{K}^* = -\mathcal{K}^*$. For a vector $x$, let $[x]_\mathcal{K}$ denote the projection of $x$ onto $\mathcal{K}$. If $A$ is the identity operator, $G$ is empty, $d = 0$, and the index set $I_{d_x} = \{ 1, \ldots, n \}$, then problem (2.6) can be reduced to

\[
\min_{z \in \mathcal{K}^*, v \in \mathbb{R}^m} \frac{1}{4} \| 2b - E^* y + z \|^2 + \langle f, y \rangle = \min_{y} \frac{1}{4} \| 2b - E^* y \|_{\mathcal{K}^*}^2 + \langle f, y \rangle,
\]

which is the formula given in [19].

### 2.4 Conic Convex Quadratic Programs

The approach in the previous section extends simply to the following conic convex quadratic program:

\[
\min_{x} \frac{1}{2} \langle x, Qx \rangle + \langle c, x \rangle \\
\text{s.t.} \quad Ex = f \\
Gx \leq h \\
x_i - d_i \in \mathcal{K}, \quad i \in I_{d_x},
\]

(2.13)
where $Q$ is symmetric positive semidefinite. The definitions for $n$, $m$, $q$, and $N_d$ are the same as that for the least squares problem.

Under some constraint qualification, we obtain its Lagrangian dual as below:

$$
\begin{align*}
\min_x \quad & \max_{y \in \mathbb{R}^m, v \geq 0, z \in \mathbb{R}^*} \langle x, Qx \rangle + 2\langle c, x \rangle + \langle y, Ex - f \rangle + \langle v, Gx - h \rangle - \langle z, \tilde{x} \rangle + \langle \tilde{z}, \tilde{d} \rangle \\
= & \max_{y \in \mathbb{R}^m, v \geq 0, z \in \mathbb{R}^*} \min_x \langle x, Qx \rangle + 2\langle c, x \rangle + \langle y, Ex - f \rangle + \langle v, Gx - h \rangle - \langle z, \tilde{x} \rangle + \langle \tilde{z}, \tilde{d} \rangle \\
= & -\min_{y \in \mathbb{R}^m, v \geq 0, z \in \mathbb{R}^*} \left\{ \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle + \max_x \left[ -\langle x, Qx \rangle - 2\langle c, x \rangle - \langle Ex, y \rangle - \langle Gx, v \rangle + \langle \tilde{x}, \tilde{z} \rangle \right] \right\}.
\end{align*}
$$

From the first order optimality condition in $x$, we obtain its dual:

$$
\begin{align*}
\min_{v \geq 0, z \in \mathbb{R}^*, u \in \mathbb{R}^N} & \langle Qu, u \rangle + \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle \\
\text{s.t.} & 2Qu + 2c + E^*y + G^*v = \left( \frac{0}{\tilde{z}} \right).
\end{align*}
$$

(2.14)

Given a triple $(y, v, \tilde{z})$, a minimum-norm solution in $x$ is

$$
x(\tilde{z}, v, y) = \frac{1}{2} Q^\dagger \left[ \left( \frac{0}{\tilde{z}} \right) - G^*v - E^*y - 2c \right].
$$

Since $Q$ is positive semidefinite, we can write $Q = A^*A$, with $A \in \mathbb{R}^{N \times N}$. Let $\{ Q^I \}_{Idx, Ids}$ denote the submatrix of $Q^I$ with row and column indices from $Idx$. As in the least squares case, the dual can be further simplified.

### 2.4.1 The full dual form.

**Theorem 2.2.** Define the Primal Range-Space Condition for quadratic program (2.13) as

$$
(I - Q^I Q) \left[ 2c + E^*y + G^*v - \left( \frac{0}{\tilde{z}} \right) \right] = 0.
$$

Under the Primal Range-Space Condition for the quadratic program, problem (2.14) is equivalent to

$$
\begin{align*}
\min_{v \geq 0, z \in \mathbb{R}^*, y \in \mathbb{R}^m} & \frac{1}{4} \left\langle E^*y + G^*v - \left( \frac{0}{\tilde{z}} \right), Q^I \left[ 2c + E^*y + G^*v - \left( \frac{0}{\tilde{z}} \right) \right] \right\rangle + \langle f, c, y \rangle + \langle h + GQ^I c, v \rangle - \langle \tilde{d}, \tilde{z} \rangle + \left\{ Q^I \right\}_{Idx, Ids} c, \tilde{z}.
\end{align*}
$$

A necessary and sufficient condition for the above problem having a finite optimal value is $f \in \mathcal{K}(E^I Q^I)$, $h \in \mathcal{K}(GQ^I) + \mathbb{R}^2_+$, $\tilde{d} \in \mathcal{K}(\{ Q^I \}_{Idx, Ids}) - \mathcal{K}^*$, and the set $\{ u \in \mathbb{R}^n : h - GQ^I u \geq 0, f - E^I Q^I u = 0, \{ Q^I \}_{Idx, Ids} u - \tilde{d} \in \mathcal{K} \}$ is non-empty.

**Proof.** The space $\mathcal{R}^N$ is a direct sum of $\mathcal{R}(Q)$ and $Q^\perp$. We have

$$
P_{Q^\perp} = I - Q^I Q = I - (A^*A)^\dagger (A^*A) \overset{(A^4)}{=} I - A^\dagger A^* A^* A \overset{(A^3)}{=} I - A^\dagger A.
$$

Applying $P_{Q^\perp}$ to both sides of the equality in the dual (2.14), we obtain the Primal Range-Space Condition for the quadratic program.

By definition (A.3) in Appendix A, we can write $Q = QQ^I Q$. Assume that the Primal Range-Space Condition for the quadratic program is satisfied. Substituting $Qu$ by $\frac{1}{2} \left[ \left( \frac{0}{\tilde{z}} \right) - 2c - E^*y - G^*v \right]$ in the objective of the dual (2.14), we get that model (2.14) is equivalent to

$$
\begin{align*}
\min_{v \geq 0, z \in \mathbb{R}^*, y \in \mathbb{R}^m} & \frac{1}{4} \left\langle E^*y + G^*v - \left( \frac{0}{\tilde{z}} \right), Q^I \left[ 2c + E^*y + G^*v - \left( \frac{0}{\tilde{z}} \right) \right] \right\rangle + \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle \\
= & \min_{v \geq 0, z \in \mathbb{R}^*, y \in \mathbb{R}^m} \frac{1}{4} \left\langle E^*y + G^*v - \left( \frac{0}{\tilde{z}} \right), Q^I \left[ E^*y + G^*v - \left( \frac{0}{\tilde{z}} \right) \right] \right\rangle + \langle f, E^I Q^I c \rangle + \langle h + GQ^I c, v \rangle - \langle \tilde{d}, \tilde{z} \rangle + \left\{ Q^I \right\}_{Idx, Ids} c, \tilde{z},
\end{align*}
$$

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which is the formula given in the theorem. By \( Q^+ = A^tA^t^* \), it is further equivalent to

\[
\min_{v \geq 0, z \in \mathbb{R}^m} \max_{u \in \mathbb{R}^n} \frac{1}{2} \langle A^t^* \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - 2c - E^t y - G^t v, u \rangle - \frac{1}{2} \|u\|^2 + \langle f, y \rangle + \langle h, v \rangle - \langle \tilde{d}, \tilde{z} \rangle
\]

\[
= \max_{u \in \mathbb{R}^n} \min_{v \geq 0, z \in \mathbb{R}^m} \frac{1}{2} \|u\|^2 + \langle f - \frac{1}{2}EA^t u, y \rangle + \langle h - \frac{1}{2}GA^t u, v \rangle - \langle \tilde{d}, \tilde{z} \rangle.
\]

Because \( R(Q^+) = R(A^t) \), \( R(\{Q^+\}_I) = R(\{A^t\}_I) \), for the inner min problem having a finite optimal value, it is necessary and sufficient that \( f \in R (EQ^1) \), \( h \in R (GQ^1) + \mathbb{R}^q \), \( \tilde{d} \in R(\{Q^+\}_I) - \mathcal{K} \), and the set \( \{u \in \mathbb{R}^N : h - GQ^1 u \geq 0, f - EQ^1 u = 0, \{Q^+\}_I u - d \in \mathcal{K} \} \) is non-empty.

\[\Box\]

### 2.4.2 The succinct dual form.

As for the least squares problem, we can further eliminating \( y \) from the above full dual form. Similar to Lemma 2.2, we have: Problem (2.13) has a unique solution in \( y \) iff \( EQ^1 E^* \) has rank \( m \). Suppose \( f \in R (EQ^1 E^*) \). Then a minimum-norm solution in \( y \) for problem (2.15) is

\[ y = (EQ^1 E^*)^+ \left\{ EQ^1 \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v - 2c \right\} - 2f \]

Substituting \( y \) by the above equation in problem (2.15), we obtain that problem (2.15) is equivalent to

\[
\min_{z \in \mathbb{R}^m, v \geq 0} \frac{1}{4} \left\{ \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v, E^* (EQ^1 E^*)^+ EQ^1 - I \right\}^{\frac{1}{2}} \left[ EQ^1 \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v \right] + \langle EQ^1 E^* v, EQ^1 - I \rangle \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v \right) + \langle f + EQ^1 c, EQ^1 \rangle \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v \rangle + \langle h + GQ^1 c, v \rangle - \langle \tilde{d}, \{Q^+\}_I c, \tilde{z} \rangle.
\]

Since

\[
(2.16) \quad \left( EQ^1 E^* \right)^+ = \left( EA^t A^t^* \right)^+ (A^t) \left( EA^t \right)^t
\]

\[
(2.17) \quad Q^* E^* \left( EQ^1 E^* \right)^+ = A^t A^t^* E^* \left( EA^t \right)^+ (A^t)^t
\]

\[
(2.18) \quad Q^* E^* \left( EQ^1 E^* \right)^+ = A^t (EA^t) \left( EA^t \right)^t
\]

\[
(2.19) \quad Q^* E^* \left( EQ^1 E^* \right)^+ = A^t \left( EA^t \right)^t
\]

\[
(2.20) \quad \left( EQ^1 E^* \right)^+ EQ^1 \left[ E^* (EQ^1 E^*)^+ EQ^1 - I \right] \left( A^t \right) \left( EQ^1 E^* \right)^+ EQ^1 - \left( EQ^1 E^* \right)^+ EQ^1 = 0,
\]

by relation (A.5), the above model can be represented as

\[
\min_{z \in \mathbb{R}^m, v \geq 0} \left\{ \| \left( EA^t \right)^t (EA^t) A^t^* - A^t^* \right\} \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v \right\}^2 + \langle A^t (EA^t)^t , \left[ f + (EA^t) A^t c \right], \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v \rangle + \langle h + GA^t A^t c, v \rangle - \langle \tilde{d}, \{A^t\}_I A^t c, \tilde{z} \rangle.
\]

And it is equivalent to

\[
\min_{z \in \mathbb{R}^m, v \geq 0} \max_{u \in \mathbb{R}^n} \frac{1}{2} \left\{ \left[ I - (EA^t)^t (EA^t) \right] A^t \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v \right\} , u \right\} + \langle A^t (EA^t)^t, \left[ f + (EA^t) A^t c \right], \left[ \begin{smallmatrix} 0 \\ z \end{smallmatrix} \right] - G^* v \rangle + \langle h + GA^t A^t c, v \rangle - \langle \tilde{d}, \tilde{z} \rangle + \langle h + GA^t \left[ I - (EA^t)^t (EA^t) \right] A^t c - \frac{1}{2} G A^t \left[ I - (EA^t)^t (EA^t) \right] u - GA^t (EA^t)^t f, v \rangle.
\]
We therefore have the following results:

**Theorem 2.3.** Let $Q = A^* A$. Suppose $f \in \mathcal{R}(EQ^T E^*)$. Let

$$y = (EA^*)^\top \left\{ A^* \left[ \begin{array}{c} 0 \\ \hat{z} \end{array} \right] - G^* v - 2c \right\} - 2(EA^*)^\top f.$$ 

Then under the Primal Range-Space Condition for the quadratic program, problem (2.14) is equivalent to (2.21). A necessary and sufficient condition for the dual (2.21) having a finite optimal value is: $\hat{d} \in \mathcal{R} \left( \{A^\top \}_{lds} \right) - \mathcal{K}, \ h \in \mathcal{R}(GA^\top) + \mathbb{R}^\top_+, \ and \ the \ set \ \{u \in \mathbb{R}^p : \{A^\top \}_{lds} [I - (EA^*)^\top (EA^*)] (u/2 - A^* c) + \{A^\top \}_{lds} (EA^*)^\top f - \hat{d} \in \mathcal{K}, \ h - GA^\top (I - (EA^*)^\top (EA^*)) (u/2 - A^* c) - GA^\top (EA^*)^\top f \geq 0 \} \ is \ non-empty.$

3 The Smooth Method

In this part, Nesterov’s smooth algorithm [22] is adapted to some special cases of least squares problem (1.8) including (1) $A^* \left[ \begin{array}{rr} E^* & G^* \left[ \begin{array}{rr} 0 \\ -I \end{array} \right] \right] \ having \ full \ column \ rank \ for \ the \ reformulation \ F(\tilde{z}, v, y)$, and (2) $P \left[ G^* \left[ \begin{array}{rr} 0 \\ -I \end{array} \right] \right] \ having \ full \ column \ rank \ for \ the \ succinct \ dual \ form \ F_y(\tilde{z}, v)$. The efficiency estimate for this modified method is $O(1/\sqrt{\tilde{e}})$. And no parameter needs to be selected.

3.1 Adapting Nesterov’s Smooth Algorithm

For the primal-dual pair (1.1)–(1.2), when $\Psi(u)$ is strongly convex, we are able to define a single-valued vector function $u_0(x) : Q_1 \mapsto Q_2$ with a constant $L_1(f)$ such that

\begin{equation}
(3.1a) \quad u_0(x) = \arg \max_{u \in Q_2} \{ \langle Hx, u \rangle - \Psi(u) \}, \ \forall x \in Q_1;
\end{equation}

\begin{equation}
(3.1b) \quad f(y) \leq f(x) + \langle \nabla \tilde{f}(x) + H^* u_0(x), y - x \rangle + \frac{1}{2} L_1(f) \| y - x \|^2, \ \forall x, y \in Q_1.
\end{equation}

Let $\Psi(u) = F(\tilde{z}, v, y)$ for model (2.5), and $\Psi(u) = F_y(\tilde{z}, v)$ for model (2.10). Then model (2.5) with the matrix $A^* \left[ \begin{array}{rr} E^* & G^* \left[ \begin{array}{rr} 0 \\ -I \end{array} \right] \right]$, having full-column rank together with function $x(\tilde{z}, v, y)$ defined in equation (2.3), and model (2.10) with the matrix $P \left[ G^* \left[ \begin{array}{rr} 0 \\ -I \end{array} \right] \right]$, having full column rank together with function $x_y(\tilde{z}, v)$ defined in equation (2.12) are such examples.

From definition (3.1a), we have

\begin{equation}
(3.2) \quad f(x) = \tilde{f}(x) + \langle Hx, u_0(x) \rangle - \Psi[u_0(x)], \ \forall x \in Q_1.
\end{equation}

And

\begin{equation}
(3.3) \quad \nabla f(x) = \nabla \tilde{f}(x) + H^* u_0(x), \ \forall x \in Q_1.
\end{equation}

For all $u \in Q_2, y \in \mathbb{R}^m, x \in Q_1$, since

$$\langle H_y, u \rangle - \Psi(u) = \langle Hx, u \rangle - \Psi(u) + \langle H^* u, y - x \rangle,$$

it follows

$$\max_{u \in Q_2} \{ \langle H_y, u \rangle - \Psi(u) \} \geq \langle Hx, u_0(x) \rangle - \Psi(u_0(x)) + \langle H^* u_0(x), y - x \rangle \geq \max_{u \in Q_2} \{ \langle Hx, u \rangle - \Psi(u) \} + \langle H^* u_0(x), y - x \rangle.$$

Further in view of the convexity of $\tilde{f}(x)$ and definition (3.3), we have

\begin{equation}
(3.4) \quad f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle, \ \forall x, y \in Q_1.
\end{equation}

Below is Nesterov’s smooth method [22].
Nesterov’s optimal scheme for smooth optimization

For $k \geq 0$, do
1. Compute $u_0(x_k)$ and $\nabla f(x_k)$.
2. Find
   \[
   \hat{x}_k = \arg \min_{x \in Q_1} \left\{ \langle \nabla f(x_k), x - x_k \rangle + \frac{L_1(f)}{2} \| x - x_k \|^2 \right\}.
   \]
3. Find
   \[
   \bar{x}_k = \arg \min_{x \in Q_1} \left\{ \frac{L_1(f)}{\sigma_1} \phi_1(x) + \sum_{i=0}^{k} \frac{i+1}{2} \left[ \langle \nabla f(x_i), x - x_i \rangle \right] \right\}.
   \]
4. Set
   \[
   x_{k+1} = \frac{2}{k+3} \hat{x}_k + \frac{k+1}{k+3} \bar{x}_k.
   \]

**Theorem 3.1.** (c.f. [22, Theorem 2]) Suppose there exist a single valued function $u(x)$ and a constant $L_1(x)$ satisfying conditions (3.1) for problem (1.1).

Let the sequences $\{x_k\}_{k=0}^{\infty}$ and $\{\hat{x}_k\}_{k=0}^{\infty}$ be generated by the above Nesterov’s optimal scheme for smooth optimization on $f$. Then for any $k \geq 0$ we have
\[
(3.5) \quad f(\hat{x}_k) \leq \min_{x \in Q_1} \left\{ \frac{4L_1(f)}{\sigma_1(k+1)(k+2)} \phi_1(x) + \sum_{i=0}^{k} \frac{2(i+1)}{(k+1)(k+2)} \left[ f(x_i) + \langle \nabla f(x_i), x - x_i \rangle \right] \right\}.
\]
Therefore,
\[
(3.6) \quad f(\hat{x}_k) - f(x^*) \leq \frac{4L_1(f) \phi_1(x^*)}{\sigma_1(k+1)(k+2)},
\]
where $x^*$ is an optimal solution to problem (1.1).

Nesterov also shows how to get an approximate solution to the dual (1.2) with little extra work. A duality gap for an approximate solution of the pair (1.1) – (1.2) can be obtained by adapting the proof of [22, Theorem 3]. For this problem, the efficiency estimate of the above scheme is $O(1/\sqrt{\epsilon})$ under the assumption of Theorem 3.1.

**Theorem 3.2.** Suppose there exist a single valued function $u_0(x)$ and a constant $L_1(x)$ satisfying conditions (3.1) for problem (1.1). After $K$ iterations of Nesterov’s optimal scheme for smooth optimization on the primal-dual problems (1.1)-(1.2), the approximate solutions
\[
(3.6) \quad \hat{x} = \bar{x}_K \in Q_1, \quad \tilde{u}_0 = \sum_{i=0}^{K} \frac{2(i+1)}{(k+1)(K+2)} u_0(x_i) \in Q_2
\]
have the following duality gap:
\[
(3.7) \quad 0 \leq f(\hat{x}) - \Psi(\tilde{u}_0) \leq \frac{4L_1(f)}{\sigma_1(k+1)(K+2)} \phi_1[\chi(\tilde{u}_0)].
\]

**Proof.** By (3.1), (3.2), (3.4), the convexity of $\Psi$, and the definition of $\tilde{u}_0$, we have
\[
\sum_{i=0}^{K} \frac{(i+1)}{(k+1)(K+2)} \left[ f(x_i) + \langle \nabla f(x_i), x - x_i \rangle \right] = \sum_{i=0}^{K} \frac{2(i+1)}{(k+1)(K+2)} \left[ \bar{f}(x_i) + \langle Hx_i, u_0(x_i) \rangle - \Psi[u_0(x_i)] + \langle \nabla \bar{f}(x_i), x - x_i \rangle + \langle H^* u_0(x_i), x - x_i \rangle \right].
\]
Since
\[ \sum_{i=0}^{K} (i+1) \langle Hx_i, u_0(x_i) \rangle + \langle H^* u_0(x_i), x - x_i \rangle = \langle Hx, \sum_{i=0}^{K} (i+1)u_0(x_i) \rangle \]
according to (3.6),
\[ \sum_{i=0}^{K} (i+1)\hat{\psi}[u_0(x_i)] \quad \text{convex} \quad \hat{f}(x_i) + \langle \nabla \hat{f}(x_i), x - x_i \rangle \leq \hat{f}(x), \]
we obtain
\[ \sum_{i=0}^{K} (i+1) \hat{f}(x_i) \leq \frac{1}{2} (K + 1)(K + 2) \left[ -\psi(\hat{u}_0) + \hat{f}(x) + \langle Hx, \hat{u}_0 \rangle \right]. \]

Substituting the above inequality into (3.5), further in view of (3.2), (1.2), and (1.6), we have
\[ f(\hat{x}) = f(\tilde{x}_k) \leq 4L_1(f) \frac{\bar{\sigma}_1(K + 1)(K + 2)}{\sigma_1(K + 1)(K + 2)} \left[ -\psi(\hat{u}_0) + \hat{f}(x) + \langle Hx, \hat{u}_0 \rangle \right], \]
from which the duality gap (3.7) follows.

3.2 The Smooth Algorithm for the Least Squares

Let the prox-function for the primal space be the squared norm \( \| \cdot \|^2 \)—its convexity parameter \( \sigma = 2 \). For problem (2.1), let \( \hat{\psi}(x) \) defined in model (1.1) be \( \| Ax - b \|^2 \). Then the strict convexity parameter for \( \hat{\psi}(x) = \| Ax - b \|^2 \) is \( \hat{\sigma} = 2\lambda_{\min}(A^*A) \).

Given a dual iterate \((\tilde{z}_k; v_k; y_k)\) for \( k = 0, \ldots \), let \( x(\tilde{x}_k; v_k; y_k) \) be defined in equation (2.3). For a scalar \( \alpha \), denote
\[ [\alpha]_+ = \begin{cases} \alpha & \alpha \geq 0 \\ 0 & \alpha < 0 \end{cases}. \]

For a vector \( u \), let \([u]_+\) denote the vector resulting from applying \([\cdot]_+\) element-wise to \( u \). Nesterov’s smooth scheme to the least squares problem can be applied directly to (2.1). This part presents how the scheme works on \( F \) and \( F_y \).

Under the Primal Range-Space Condition  Assume that model (2.2) satisfies the Primal Range-Space Condition. The constant \( L_1(f) \) defined in (3.1) for reformulation (2.6) is
\[ L(F) \equiv \frac{1}{2} \left\| A^* \left[ \begin{array}{c} 0 \\ -G^* \end{array} \right] - E^* \right\|_2^2. \]
Nesterov’s Smooth Method for the Least Squares Problem

Let \( \tilde{z}_0 = 0, v_0 = 0, y_0 = 0, x_0 = A^\dagger b \). For \( k \geq 0 \), do

1. \[
\begin{align*}
\tilde{z}_k &= \left[ z_k - \frac{1}{L(F)} \nabla_z F(z_k, v_k, y_k) \right] \mathcal{K}^* \\
v_k &= \left[ v_k - \frac{1}{L(F)} \nabla_v F(z_k, v_k, y_k) \right] + \\
y_k &= y_k - \frac{1}{L(F)} \nabla_y F(z_k, v_k, y_k) \\
\end{align*}
\]

2. \[
\begin{align*}
\tilde{z}_{k+1} &= \frac{k+1}{k+3} \tilde{z}_k + \frac{1}{k+3} \left[ - \frac{1}{L(F)} \sum_{i=0}^{k} (i+1) \nabla_z F(z_i, v_i, y_i) \right] \mathcal{K}^* \\
v_{k+1} &= \frac{k+1}{k+3} v_k + \frac{1}{k+3} \left[ - \frac{1}{L(F)} \sum_{i=0}^{k} (i+1) \nabla_v F(z_i, v_i, y_i) \right] + \\
y_{k+1} &= y_k - \frac{1}{L(F)} \sum_{i=0}^{k} (i+1) \nabla_y F(z_i, v_i, y_i) \\
\end{align*}
\]

3. \( x_{k+1} = x(\tilde{z}_{k+1}, v_{k+1}, y_{k+1}) \).

The approximate primal solution after \( K \) iterations is

\[
\hat{x} = \sum_{i=0}^{K} \frac{2(i+1)}{(K+1)(K+2)} x_i.
\]

The succinct dual form. When the assumptions in Theorem 2.1 are satisfied, problem (1.8) can be solved approximately through reformulation (2.10). The Lipschitz constant for \( F_y \) is

\[
L(F_y) = \frac{1}{2} \left\| I - (EA^\dagger)^* (EA^\dagger)^* A^\dagger \begin{bmatrix} 0 \\ I \end{bmatrix} \right\|_{2}^2.
\]

Let \((v; \tilde{z})\) represent the vector generated by stacking the vectors \( v \) and \( \tilde{z} \) together. Substituting (2.7) into (2.1), we obtain the operator relating the primal variable \( x \) to the dual variables \((v; \tilde{z})\) as

\[
H = \left( I - E^* (EA)^* A^* \right) \begin{bmatrix} 0 \\ I \end{bmatrix} - G^*.
\]

Let \( x_y(v; \tilde{z}) \) be as defined in (2.12). Nesterov’s smooth scheme applied to problem (2.10) works as follows:
Nesterov’s Smooth Method for the Least Squares Problem - no γ

Let \( \bar{x}_0 = 0, v_0 = 0, x_0 = x_f(\bar{x}_0, v_0) \). For \( k \geq 0 \), do

1. \[
\bar{x}_k = \frac{1}{L(f_\delta)} \nabla \bar{f}_\delta(\bar{x}_k, v_k)
\]

2. \[
v_k = \frac{1}{L(f_\delta)} \nabla v \bar{f}_\delta(\bar{x}_k, v_k)
\]

3. \[
x_{k+1} = x_f(\bar{x}_{k+1}, v_{k+1}).
\]

The approximate primal solution after \( K \) iterations is

\[
\hat{x} = \sum_{i=0}^{K} \frac{2(i+1)}{(K+1)(K+2)} x_i.
\]

4 Excessive Gap Method

This section has three parts: §§4.1 extends Nesterov’s excessive gap method [21] to general cases of problems (1.1)–(1.2). §§4.2 applies the modified method to the least squares problem (1.8). §§4.3 adapts Nesterov’s Excessive Gap method to the least squares problem (1.8).

For problems (1.1)–(1.2), denote \( D_x \equiv \max x \phi_1(x(\tilde{x}_k)) \) and \( D_\delta \equiv \max x \phi(\tilde{x}_k) \), where \( (\tilde{x}_k, \tilde{u}_k) \) are iterates generated by the excessive gap method. Let \( L_1(f_{\mu_2}) \) and \( L_2(\mu_\gamma) \) be the Lipschitz constants of \( f_{\mu_2} \) and \( \mu_\gamma \) respectively. Since both \( f_{\mu_2} \) and \( \mu_\gamma \) have Lipschitz continuous gradients, we have

\[
L_1(f_{\mu_2}) = L_1(\hat{f}) + \frac{||H||^2_2}{\mu_2 \sigma}, \quad L_2(\mu_\gamma) = L_2(\hat{\psi}) + \frac{||H||^2_2}{\mu_1 \sigma_1}.
\]

Let \( u_{\mu_2, \gamma}(x) \) denote the optimal solution of \( f_{\mu_2, \gamma}(x) \) in \( u \) and let \( x_{\mu_1, \gamma}(u) \) denote the optimal solution of \( \psi_{\mu_1, \gamma}(u) \) in \( x \).

4.1 Extension of Nesterov’s Excessive Gap Method 1

Based on the excessive gap technique, Nesterov designed method [21] for problems (1.1)–(1.2) with \( L_1(\hat{f}) = L_2(\hat{\psi}) = 0 \). In this part, Nesterov’s Excessive Gap Method 1 will be extended to problems (1.1)–(1.2) having \( L_1(\hat{f}) \geq 0 \) and \( L_2(\hat{\psi}) \geq 0 \). To this end, smoothness parameters different from those in [21] will be used; below are two possible choices of smoothness parameters.

Denote the parameters \( \lambda_1 \) and \( \lambda_2 \) as that in [21]:

\[
\alpha_k = \frac{2}{k + 2}, \quad k \geq -1; \quad \alpha_{k+1} = (1 - \tau_k) \alpha_{k-1}, \quad k \geq 0;
\]

\[
k = 2l : \quad \lambda_{1,k} = \alpha_{k-1}, \quad \lambda_{2,k} = \alpha_k;
\]

\[
k = 2l + 1 : \quad \lambda_{1,k} = \alpha_k, \quad \lambda_{2,k} = \alpha_{k-1}.
\]

Define
1. Parameter set 1:

\[(4.3) \quad \mu_1 = \lambda_1 \left( \|H\|_2 \sqrt{\frac{D_u}{\sigma_1 D_x}} + L_1(\hat{f})/\sigma_1 \right), \quad \mu_2 = \lambda_2 \left( \|H\|_2 \sqrt{\frac{D_u}{\sigma_1 D_x}} + L_2(\hat{\psi})/\sigma \right).\]

2. Parameter set 2:

\[(4.4) \quad t = \max \left( \frac{L_1(\hat{f})}{2} \sqrt{\frac{\sigma D_u}{\sigma_1 D_x}} + \sqrt{\frac{L_1^2(\hat{f})}{4} \frac{\sigma D_x}{\sigma_1 D_x}} + \|H\|_2^2, \quad \frac{L_2(\hat{\psi})}{3} \sqrt{\frac{\sigma_1 D_u}{\sigma D_x}} + \sqrt{\frac{L_1^2(\hat{\psi})}{9} \frac{\sigma_1 D_u}{\sigma D_x}} + \|H\|_2^2 \right), \quad \mu_1 = \lambda_1 t \sqrt{\frac{D_u}{\sigma_1 D_x}}, \quad \mu_2 = \lambda_2 t \sqrt{\frac{D_x}{\sigma_1 D_u}}.\]

When \(L_1(\hat{f}) = L_2(\hat{\psi}) = 0\), the above smoothness parameters are identical to that used by Nesterov [21].

We can minimize the duality gap \(f(x) - \psi(u)\) by METHOD 1 of [21] with these parameter sets. Define

\[
T_{\mu_2}(x) = \arg \min_{y \in \Omega_1} \left\{ \langle \nabla f_{\mu_2}(x), y - x \rangle + \frac{L_1(f_{\mu_2})}{2} \|y-x\|^2 \right\}.
\]

**Modified Nesterov’s Excessive Gap METHOD 1**

1. Set \(\mu_{1,0}\) and \(\mu_{2,0}\) according to (4.3) or (4.4). Let \(\bar{x}_0 = T_{\mu_{2,0}}(\bar{x})\), \(\bar{u}_0 = u_{\mu_{2,0}}(\bar{x})\).

2. \(k \geq 0\)

(a) Set \(\tau_k = \frac{2}{k+3}\).

(b) If \(k\) is even, let

\[
\begin{align*}
\mu_{1,k+1} &= (1 - \tau_k)\mu_{1,k}, \quad \hat{x}_k = (1 - \tau_k)\bar{x}_k + \tau_k u_{\mu_1,k}(\bar{u}_k), \\
\bar{u}_{k+1} &= (1 - \tau_k)\bar{u}_k + \tau_k u_{\mu_2,k}(\hat{x}_k), \quad \bar{x}_{k+1} = T_{\mu_2,k}(\hat{x}_k).
\end{align*}
\]

(c) If \(k\) is odd, generate \((\bar{x}_{k+1}, \bar{u}_{k+1})\) from \((\bar{x}_k, \bar{u}_k)\) using the symmetric dual variant of (4.5).

To prove the convergence of the above modified METHOD 1 with the proposed smoothness parameters, we need to verify that the excessive gap condition (1.5) holds for each iterate. By [21, Theorem 4.2], when the pair \((\bar{x}_k, \bar{u}_k)\) satisfies the excessive gap condition (1.5) with smoothness parameters \(\mu_{1,k}\) and \(\mu_{2,k}\), if

\[
\frac{\tau_k^2}{1 - \tau_k} \leq \frac{\mu_{1,k} \sigma_1}{L_1(f_{\mu_2,k})},
\]

one can conclude that iterate \((\bar{x}_{k+1}, \bar{u}_{k+1})\) generated by relations (4.5) maintains the excessive gap condition (1.5) with smoothness parameters \(\mu_{1,k+1}\) and \(\mu_{2,k+1}\). Therefore, we will prove condition (4.6) for iterations \(k \geq 0\).

**Lemma 4.1.** Suppose \(L_1(\hat{f}) \geq 0\) and \(L_2(\hat{\psi}) \geq 0\). At iteration \(k \geq 0\) of METHOD 1, if smoothness parameters are defined by (4.3) or (4.4), then

\[
\frac{\tau_k^2}{1 - \tau_k} \leq \frac{\mu_{1,k} \sigma_1}{L_1(f_{\mu_2,k})} (k \geq 0 \text{ even}), \quad \frac{\tau_k^2}{1 - \tau_k} \leq \frac{\mu_{2,k} \sigma}{L_2(\psi_{\mu_1,k})} (k \geq 1 \text{ odd}).
\]

**Proof.** By definition (4.2), one concludes that the sequences \(\{\lambda_{1,k}\}_{k=0}^\infty\) and \(\{\lambda_{2,k}\}_{k=0}^\infty\) are non-increasing, and

\[
\lambda_{1,k} \leq \frac{2}{k+1} (k \geq 1), \quad \lambda_{2,k} \leq 1 (k \geq 0); \quad \tau_k = \frac{2}{k+3}.
\]

Hence,

\[
\frac{\tau_k^2}{1 - \tau_k} = \frac{4}{(k+1)(k+3)} \leq \frac{4}{(k+1)(k+2)} = \alpha_k \alpha_{k-1} = \lambda_{1,k} \lambda_{2,k}.
\]
Let us first prove the lemma under smoothness parameter set 1 defined in equations (4.3).

- When \( k \geq 0 \) is even, since \( \lambda_{1,k} \leq 1 \), we have

\[
\lambda_{1,k} \lambda_{2,k} \leq \frac{\lambda_{1,k} \lambda_{2,k} \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_1(\hat{f}) \right) \cdot \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_2(\hat{\psi}) \right)}{\lambda_{2,k} L_1(\hat{f}) \cdot \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_2(\hat{\psi}) \right) + \left\| H \right\|_2^2}
\]

Therefore,

\[
(4.8) \quad \lambda_{1,k} \lambda_{2,k} \leq \frac{\lambda_{1,k} \lambda_{2,k} \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_1(\hat{f}) \right) \cdot \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_2(\hat{\psi}) \right)}{L_1(\hat{f}) + \lambda_{2,k} \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_2(\hat{\psi}) \right) + \left\| H \right\|_2^2}
\]

Further in view of equation (4.7), we have

\[
\frac{\mu_1, k \sigma_1}{L_1(f_{\psi_{1, k}})}.
\]

- Symmetrically, for \( k \geq 1 \) being odd, since \( \lambda_{1,k} \leq \frac{2}{\lambda_1} \), we have

\[
\left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_1(\hat{f}) \right) \cdot \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_2(\hat{\psi}) \right) \geq \lambda_{1,k} \frac{L_2(\hat{\psi})}{\lambda_{1,k} \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_2(\hat{\psi}) \right) + \left\| H \right\|_2^2}
\]

Therefore,

\[
(4.3), (4.1) \quad \lambda_{1,k} \lambda_{2,k} \leq \frac{\lambda_{1,k} \lambda_{2,k} \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_1(\hat{f}) \right) \cdot \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_2(\hat{\psi}) \right)}{L_2(\hat{\psi}) + \lambda_{1,k} \left( \left\| H \right\|_2 \sqrt{\frac{\sigma_1 D_u}{\sigma_1 D_x}} + L_2(\hat{\psi}) \right) + \left\| H \right\|_2^2}
\]

Further in view of equation (4.7), we have

\[
\frac{\mu_2, k \sigma}{L_2(\psi_{\mu_{1, k}})}.
\]

- Next let us prove Lemma 4.1 for smoothness parameters chosen according to parameter set 2 in equations (4.4).

- For \( k \geq 0 \) being even, from the definition of \( t \) in equations (4.4), we have

\[
t^2 - \mu_1, k \sigma_1 \leq \mu_2, k \sigma_1.
\]
Further by \( \lambda_{2,k} \leq 1 \), we have

\[
\lambda_{1,k} \lambda_{2,k} \leq \frac{\lambda_{1,k} \lambda_{2,k} t^2}{tL_1(\hat{f}) \lambda_{2,k} \sqrt{\frac{\sigma_D}{\sigma_1 \sigma_u}} + \|H\|^2} \quad (4.9)
\]

In view of equation (4.7), we obtain

\[
\frac{\tau_k^2}{1 - \tau_k} \leq \frac{\mu_{1,k} \sigma_1}{L_1(f_{p_{2,k}})}.
\]

- Symmetrically, for \( k \geq 1 \) being odd, since \( \lambda_{1,k} \leq \frac{2}{3} \) and

\[
t^2 \geq \frac{2}{3} tL_2(\hat{f}) \sqrt{\frac{\sigma_D}{\sigma_1 \sigma_u}} + \|H\|^2 \geq \lambda_{1,k} tL_2(\hat{f}) \sqrt{\frac{\sigma_D}{\sigma_1 \sigma_u}} + \|H\|^2,
\]

we have

\[
\lambda_{1,k} \lambda_{2,k} \leq \frac{\lambda_{1,k} \lambda_{2,k} t^2}{tL_2(\hat{f}) \lambda_{1,k} \sqrt{\frac{\sigma_D}{\sigma_1 \sigma_u}} + \|H\|^2} = \frac{\lambda_{2,k} t}{L_2(\hat{f}) \lambda_{1,k} \sqrt{\frac{\sigma_D}{\sigma_1 \sigma_u}}} \frac{\mu_{2,k} \sigma_1}{L_1(f_{p_{2,k}})} \quad (4.4), (4.1).
\]

Further in accordance with equation (4.7), we have

\[
\frac{\tau_k^2}{1 - \tau_k} \leq \frac{\mu_{2,k} \sigma}{L_2(\hat{f}_{\psi_{1,k}})}.
\]

To complete the proof for the convergence of the method, we also need to verify that the starting points satisfy the excessive gap condition (1.5): For \( k = 0 \), by [21, Lemma 4.1], since \( \tilde{x}_0 = T_{\mu_{1,k}}(\hat{x}) \) and \( \tilde{u}_0 = u_{\mu_{2,k}}(\hat{x}) \), the excessive gap condition is satisfied if

\[
(4.10) \quad \mu_{1,0} \geq \frac{1}{\sigma_1} L_1(f_{p_{2,0}}).
\]

**Lemma 4.2.** If smoothness parameters are chosen according to (4.3) or (4.4), then condition (4.10) is satisfied.

**Proof.** By sequence (4.2), we have

\[
\lambda_{1,0} = 2, \quad \lambda_{2,0} = 1.
\]

Further in view of inequality (4.8) for smoothness parameter set 1 defined in equations (4.3), or in view of inequality (4.9) for smoothness parameter set 2 defined in equations (4.4), we have

\[
2 = \lambda_{1,0} \lambda_{2,0} \leq \frac{\mu_{1,0} \sigma_1}{L_1(f_{p_{2,0}})},
\]

which implies inequality (4.10).

**Theorem 4.1.** Let \( \{(\tilde{x}_k, \tilde{u}_k)\}_{k=0}^\infty \) be generated by the Modified Nesterov’s Excessive Gap METHOD 1. Then each pair satisfies the excessive gap condition (1.5). Therefore,

- if the smoothness parameters are chosen according to (4.3), then

\[
f(\tilde{x}_k) - \psi(\tilde{u}_k) \leq \frac{2}{k+1} \left(2\|H\|^2 \sqrt{\frac{D_x D_u}{\sigma_1}} + L_1(\hat{f}) D_x / \sigma_1 + L_2(\hat{f}) D_u / \sigma_1\right);
\]
if the smoothness parameters are chosen according to (4.4), let

\[
t = \max \left( \frac{L_1(f)}{2} \sqrt{\frac{\sigma D_x}{\sigma_1 D_u}} + \sqrt{\frac{L_2(f)^2 \sigma D_x}{\sigma_1 D_u}} + \frac{\|H\|_2^2}{2}, \frac{L_2(\psi)\sigma_1 D_u}{3} \sqrt{\frac{\sigma_1 D_x}{\sigma D_x}} + \sqrt{\frac{L_2(\psi)^2 \sigma_1 D_u}{9 \sigma D_x}} + \frac{\|H\|_2^2}{2} \right),
\]

then

\[
f(\bar{x}_k) - \psi(\bar{u}_k) \leq \frac{4}{k+1} \sqrt{D_x D_u \sigma^2}.
\]

Proof. The excessive gap condition for each iterate is established in Lemmas (4.1, 4.2). Further in accordance to inequality (1.7), we can bound the duality gap by using the representations of \(\mu_1\) and \(\mu_2\) in (4.3) and (4.4) respectively, and \(\lambda_1, k\) \((4.2)\) \(\lambda_2, k\) \((4.2)\)

\[
\lambda_1, k \leq \frac{2}{k+1}, \quad \lambda_2, k \leq \frac{2}{k+1}.
\]

Hence,

\[
0 \leq f(\bar{x}_k) - \psi(\bar{u}_k) \leq \mu_1 D_x + \mu_2 D_u.
\]

Parameter set 1:

\[
\mu_1, k D_x + \mu_2, k D_u \leq \lambda_1, k \left( \|H\|_2 \sqrt{\frac{D_x D_u}{\sigma_1}} + L_1(f) / \sigma_1 D_x \right) + \lambda_2, k \left( \|H\|_2 \sqrt{\frac{D_x D_u}{\sigma_1}} + L_2(\psi) / \sigma D_u \right)
\]

\[
 \leq \frac{2}{k+1} \left( 2 \|H\|_2 \sqrt{\frac{D_x D_u}{\sigma_1}} + L_1(f) / \sigma_1 D_x \right) + L_1(f) D_x / \sigma_1 + L_2(\psi) D_u / \sigma
\]

Parameter set 2:

\[
\mu_1, k D_x + \mu_2, k D_u \leq \lambda_1, k \sqrt{\frac{D_x D_u}{\sigma_1}} + \lambda_2, k \sqrt{\frac{D_x D_u}{\sigma_1}} \leq \frac{4}{k+1} \sqrt{D_x D_u \sigma^2}.
\]

4.2 Modified Nesterov’s Excessive Gap METHOD 1 for the Least Squares Problem

This part applies the above Modified Nesterov’s Excessive Gap METHOD 1 to the least squares problem in the form (2.1): \(\min_{(x, y, v) \in R^n, v \geq 0} \{ \langle f, y \rangle + \langle h, v \rangle - \langle d, \bar{z} \rangle + \max_x [-\|Ax - b\|^2 - \langle Ex, y \rangle - \langle Gx, v \rangle + \langle \bar{x}, \bar{z} \rangle] \}.\) Let \(\phi(x) = \|x\|^2, \phi_1(\bar{x}, v, y) = \|(\bar{x}, v, y)\|^2.\) The matrix relating the primal and dual variables is

\[
\begin{pmatrix}
0 \\
I
\end{pmatrix}
\begin{pmatrix}
-G^* \\
-E^*
\end{pmatrix}
\]

Denote

\[
\lambda \equiv \left\| \begin{pmatrix}
0 \\
I
\end{pmatrix}
\begin{pmatrix}
-G^* \\
-E^*
\end{pmatrix}
\right\|_2^2.
\]
Modified Nesterov’s Excessive Gap METHOD 1 for the Least Squares Problem

1. Set $\mu_{1,0}$ and $\mu_{2,0}$ according to (4.3) or (4.4). Let

$$x_0 = (A^*A + \mu_{2,0}I)^{-1}A^*b,$$

$$\tilde{z}_0 = \frac{2\mu_{2,0}}{\lambda} \langle \bar{d} - \tilde{x}_0 \rangle_{\mathcal{X}^*}, \quad v_0 = \frac{2\mu_{2,0}}{\lambda} \langle Gx_0 - h \rangle_+, \quad y_0 = \frac{2\mu_{2,0}}{\lambda} \langle Ex_0 - f \rangle.$$

2. For $k \geq 0$

   (a) Set $\tau_k = \frac{2}{k+3}$.

   (b) If $k$ is even, let $\mu_{1,k+1} = (1 - \tau_k)\mu_{1,k}$. Let

   $$\hat{y}_k = (1 - \tau_k)y_k + \frac{\tau_k}{2\mu_{1,k}} (E\hat{x}_k - f), \quad \hat{v}_k = (1 - \tau_k)v_k + \frac{\tau_k}{2\mu_{1,k}} [G\hat{x}_k - h]_+, \quad \hat{\tilde{z}}_k = \frac{1}{2} (A^*A + \mu_{2,k}I)^{-1} \left( 2A^*b - E^*\hat{y}_k - G^*\hat{v}_k + \hat{\tilde{z}} \right),$$

   $$x_{k+1} = (1 - \tau_k)x_k + \tau_k \hat{y}_{\mu_{1,k}}, \quad y_{k+1} = \hat{y}_k + \frac{2\mu_{2,k}}{\lambda} (E\hat{y}_{\mu_{2,k}} - f), \quad v_{k+1} = \left[ \hat{v}_k + \frac{2\mu_{2,k}}{\lambda} \langle G\hat{y}_{\mu_{2,k}} - h \rangle_{\mathcal{X}^*} \right], \quad \tilde{z}_{k+1} = \left[ \hat{\tilde{z}} + \frac{2\mu_{2,k}}{\lambda} (\bar{d} - \hat{\tilde{x}}_{\mu_{2,k}}) \right]_{\mathcal{X}^*}.$$

   (c) If $k$ is odd, let $\mu_{2,k+1} = (1 - \tau_k)\mu_{2,k}$. Let

   $$\hat{x}_k = (1 - \tau_k)x_k + \frac{\tau_k}{2} (A^*A + \mu_{2,k}I)^{-1} (2A^*b - E^*y_k - G^*v_k + \tilde{z}_k),$$

   $$\hat{y}_{\mu_{1,k}} = \frac{1}{2\mu_{1,k}} (E\hat{x}_k - f), \quad \hat{v}_{\mu_{1,k}} = \frac{1}{2\mu_{1,k}} [G\hat{x}_k - h]_+, \quad \hat{\tilde{z}}_{\mu_{1,k}} = \frac{1}{2\mu_{1,k}} [\bar{d} - \hat{x}_k]_{\mathcal{X}^*},$$

   $$\tilde{x}_{k+1} = \tilde{x}_k - \left[ 2A^*(A\tilde{x} - b) + E^*\tilde{y}_{\mu_{1,k}} + G^*\tilde{v}_{\mu_{1,k}} - \tilde{z}_{\mu_{1,k}} \right] / \left( 2\|A\|_2^2 + \frac{\lambda}{2\mu_{1,k}} \right).$$

4.3 Nesterov’s Excessive Gap METHOD 2

Nesterov’s Excessive Gap METHOD 2 [21] is designed for the primal-dual problems (1.1)—(1.2) with $f$ being strongly convex. The excessive gap condition is maintained at each iteration of METHOD 2:

(4.11) $$\tilde{x} \in Q_1, \; \tilde{u} \in Q_2, \; f(\tilde{x}) \leq \psi_{\mu_1}(\tilde{u}).$$

Since for any $\mu \geq 0$,

(4.12) $$\psi_{\mu}(\tilde{u}) \overset{(1.4)}{=} -\tilde{\psi}(\tilde{u}) + \left\{ \langle Hx(\tilde{u}), \tilde{u} \rangle + \hat{f}(x(\tilde{u})) + \mu \tilde{\phi}_1(x(\tilde{u})) \right\} \overset{(1.6)}{=} \psi(\tilde{u}) + \mu \phi_1(x(\tilde{u})),$$

we obtain that for any pair $(\tilde{x}, \tilde{u}) \in Q_1 \times Q_2$ satisfying (4.11), their duality gap can be bounded as below:

$$0 \leq f(\tilde{x}) - \psi(\tilde{u}) \leq \mu \phi_1(x(\tilde{u})).$$

As for the smooth method, under construction (3.1) we can extend Nesterov’s Excessive Gap METHOD 2 [21] to problems (1.1)-(1.2). Denote the adjoint gradient mapping

$$V(x) = \arg \min_{v \in Q_1} \left\{ \langle \nabla f(x), v - x \rangle + \frac{1}{2} L_4(f)\|v - x\|^2 \right\}.$$

Nesterov’s Excessive Gap METHOD 2
1. Set $\mu_{1,0} = \frac{2}{\sigma_1} L_1(f), \bar{x}_0 = V(\bar{x})$ and $\bar{u}_0 = u_0(\bar{x})$.  
2. For $k \geq 0$ iterate:  
   (a) Set $\tau_k = \frac{2}{k+1}$ and $\bar{x}_k = (1 - \tau_k) \bar{x}_k + \tau_k \mu_{1,k-1}(\bar{u}_k)$.
   (b) Update $\mu_{1,k+1} = (1 - \tau_k) \mu_{1,k}, \bar{u}_{k+1} = (1 - \tau_k) \bar{u}_k + \tau_k u_0(\bar{x}_k), \bar{x}_{k+1} = V(\bar{x}_k)$.

Below is a convergence result of the above symmetric version of [21, METHOD 2].

**Lemma 4.3.** Let $u_0(x)$ be a single-valued function satisfying (3.1). Then the excessive gap condition (4.11) is valid for $\mu_1 \geq \frac{1}{\sigma_1} L_1(f), \bar{x} = V(\bar{x}), \bar{u} = u_0(\bar{x})$.

**Proof.** The proof is symmetric to the proof for [21, Lemma 7.4]. \(\square\)

Similarly, we can obtain a symmetric version of [21, Theorem 7.5] below. The detailed proof is omitted here, since it is straightforward.

**Theorem 4.2.** Suppose that there exists a single-valued function $u_0(x)$ satisfying (3.1). Let $(\bar{x}, \bar{u}) \in Q_1 \times Q_2$ satisfy the excessive gap condition (4.11) for some $\mu_1 > 0$. Let us fix $\tau \in (0,1)$ and choose

$$\mu_1^+ = (1 - \tau) \mu_1, \quad \bar{x} = (1 - \tau) \bar{x} + \tau \mu_1(\bar{u}), \quad \bar{u}_+ = (1 - \tau) \bar{u} + \tau u_0(\bar{x}), \quad \bar{x}_+ = V(\bar{x}).$$

Then $(\bar{x}_+, \bar{u}_+)$ maintains the excessive gap condition (4.11) with smoothness parameter $\mu_1^+$, provided that

$$\frac{\tau^2}{1 - \tau} \leq \frac{\mu_1 \sigma_1}{L_1(f)}.$$  

We then have the following symmetric version of [21, Theorem 7.6].

**Theorem 4.3.** Assume that there is a single-valued function $u_0(x)$ satisfying (3.1) for problem (1.1). Then the pairs $(\bar{x}_k, \bar{u}_k)$ generated by the above Nesterov’s Excessive Gap METHOD 2 satisfy the following inequality:

$$0 \leq f(\bar{x}_k) - \psi(\bar{u}_k) \leq \frac{4 L_1(f) \psi(\bar{u}_k)}{(k+1)(k+2) \sigma_1}.$$  

**Proof.** Lemma 4.3 and Theorem 4.2 guarantee that each iterate satisfies the excessive gap condition (1.5). From the algorithm we have

$$\mu_{1,k} = \frac{4 L_1(f)}{(k+1)(k+2) \sigma_1}.$$  

Further in view of relation (1.7), we obtain the theorem. \(\square\)

4.4 Nesterov’s Excessive Gap METHOD 2 for the Least Squares Problem

Nesterov’s Excessive Gap METHOD 2 can be applied directly to (2.1). This part presents how the scheme works on reformulations $F(\bar{z}, v, y)$ (2.6) and $F_2(y)$ (2.10).

Let $\phi_1(\bar{z}, v, y) = \|(\bar{z}, v, y)\|^2, \phi(x) = \|x\|^2$. Then $\sigma = \sigma_1 = 2$.

**Under the Primal Range-Space Condition** Suppose that problem (1.8) satisfies the Primal Range-Space Condition. The Lipschitz constant for function $F$ given in (2.6) is

$$L(F) = \frac{1}{2} \left\| A^* \begin{bmatrix} 0 \\ I \end{bmatrix} - G^* - E^* \right\|^2.$$  

Nesterov’s Excessive Gap METHOD 2 applied to the dual (2.6) works as follows:
Nesterov’s Excessive Gap METHOD 2 for the Least Squares Problem

1. Set $\mu_{1,0} = L(F)$, $\bar{x}_0 = A^i b$, $\bar{v}_0 = \frac{1}{L(F)} [\bar{d} - \bar{x}_0]_+$. $\bar{v}_0 = \frac{1}{L(F)} [G\bar{x}_0 - h]_+$, $\bar{y}_0 = \frac{1}{L(F)} [E\bar{x}_0 - f]$.

2. For $k \geq 0$ iterate:

   (a) Set
   
   \[
   \hat{z}_k = \frac{k + 1}{k + 3} \hat{z}_k + \frac{1}{\mu_{1,k}(k + 3)} [\bar{d} - \bar{x}_k]_+,
   \]
   
   \[
   \hat{v}_k = \frac{k + 1}{k + 3} \hat{v}_k + \frac{1}{\mu_{1,k}(k + 3)} [G\hat{x}_k - h]_+,
   \]
   
   \[
   \hat{y}_k = \frac{k + 1}{k + 3} \hat{y}_k + \frac{1}{\mu_{1,k}(k + 3)} (E\hat{x}_k - f).
   \]

   (b) Update $\mu_{1,k+1} = \frac{k + 1}{k + 3} \mu_{1,k}$,

   \[
   \tilde{x}_{k+1} = \frac{k + 1}{k + 3} \tilde{x}_k + \frac{2}{k + 3} \chi(\hat{z}_k, \hat{v}_k, \hat{y}_k),
   \]
   
   \[
   \tilde{\xi}_{k+1} = \left[ \tilde{z}_k - \frac{1}{L(F)} \nabla F(\hat{z}_k, \hat{v}_k, \hat{y}_k) \right]_+,
   \]
   
   \[
   \tilde{\nu}_{k+1} = \left[ \tilde{v}_k - \frac{1}{L(F)} \nabla F(\hat{z}_k, \hat{v}_k, \hat{y}_k) \right]_+,
   \]
   
   \[
   \tilde{\eta}_{k+1} = \tilde{y}_k - \frac{1}{L(F)} \nabla F(\hat{z}_k, \hat{v}_k, \hat{y}_k).
   \]

The succinct dual form. We can also apply Nesterov’s excessive gap METHOD 2 to the succinct dual form using $x_i$ defined in (2.12) and replacing $y$ in the Lagrange (2.1) by the right-hand-side of equation (2.7).

5 Numerical Examples

Advantages of fast gradient methods over second-order methods. Compared with second-order methods, the approach proposed in this paper has the following advantages:

1. The computational complexity of each iterations is low: only the gradients, but not the Hessians, are used. This is more and more advantageous with the increase in the numbers of constraints. In addition, the computation of each component $\tilde{z}_{k+1}^{(i)}$ for $i \in Idx$ is separable and can be carried out in parallel.

2. Primal-dual nondegeneracy and strict complementarity are not conditions for the convergence of the method. Numerical examples in §§5.3 shows that this method can get more accurate solutions than interior point method do on problem instances with possibly dual degeneracy.

3. The methods can accommodate inequality constraints and free variables more easily.

4. The cone $\mathcal{K}$ is more general than those admitting proper barriers for interior-point methods. It is only assumed that $\mathcal{K}$ is closed and convex, and that the projection onto $\mathcal{K}^+$ is easy to compute. The operator $A$ is also more general.

Below are some numerical test results on the methods discussed above. All the examples were conducted in a windows MATLAB 7.11 environment on a 3.4GHz Intel Core i7 PC with 8GB RAM.

The examples are least squares problems over the cone of positive semidefinite symmetric matrices. Conversion between vectors and matrices in the implementation is through the vec2symmat and symmat2vec functions written by MathWorks Support Team. Let $\mathcal{K} = \otimes_{i \in Idx} \mathcal{S}^{n_i}_+$ be the cone of real positive semidefinite symmetric matrices, where $n_i$ is the dimension of $\mathcal{X}_i$. Then $\mathcal{K}^+ = \mathcal{K}$. Let $x_i = U_i \Lambda_i U_i^*$ be the eigenvalue decomposition of the real symmetric matrix $x_i$, where $U_i$ is unitary.
orthogonal and $\Lambda_i$ is a diagonal matrix with eigenvalues of $x_i$ on its diagonal. Then the projection of $x_i$ onto the cone of real positive semidefinite symmetric matrices is

$$[x_i]_{\mathbb{S}^n_+} = U[\Lambda_i]_+U^t.$$

The proposed approach in this paper can be extended to sparse problems using sparse matrix computation tools such as Saad’s SPARKit and sparse matrix functions in MATLAB; as the focus of this paper is on algorithm development, sparsity issues will not be further explored in the paper. In §§5.1, Nesterov’s smooth method and Nesterov’s Excessive Gap METHOD 2 on reformulations $F(\tilde{z}, v, y)$ (2.6) and $F_i(\tilde{z}, v)$ (2.10) converge faster than Nesterov’s smooth method on the original least squares model for the example. In §§5.2, the modified Nesterov’s Excessive Gap METHOD 1 on reformulation $F(\tilde{z}, v, y)$ is compared with SDPT3-4.0 [31] and Sedumi-1.3 [29] through the YALMIP-20120609 interface [18] on the H-norm nearest correlation matrix computation problem. YALMIP is a modelling language that models problems into standard semidefinite second-order cone programs. This example shows that with less than 3% of the time used by SDPT3 the proposed approach can obtain a relatively accurate solution on problem instances that are too large for Sedumi. In §§5.3 the examples show that Nesterov’s smooth method and Nesterov’s Excessive Gap METHOD 2 on reformulation $F(\tilde{z}, v, y)$ can obtain accurate solutions on some possibly degenerate problems that cannot be solved by SDPT3 or Sedumi through the YALMIP-20120609 interface.

5.1 Progress of the Algorithms

The first example is a randomly generated semidefinite least squares problem. It has 4 blocks, with dimensions $(7, 15, 8, 16)$. Each block of the solution $x$ was created as follows: for block $i$ ($i = 1, \ldots, 4$), a random matrix $T$ of size $N_i$ by $N_i$ with each entry uniformly distributed between $(-0.5, 0.5)$ was first generated by the rand function of MATLAB, and the solution of $x$ at block $i$ was then obtained as $x_i = TT^t$. The operators $A \in \mathbb{R}^{320 \times 320}$, $E \in \mathbb{R}^{10 \times 320}$, $G \in \mathbb{R}^{5 \times 320}$, and the parameter $b$ are generated in the same way except with different scaling. The vector $f = Ex$, and the vector $h$ equals to $Gx$ plus a random number uniformly distributed in $[1, 2]$. The operator $A$ is square and is likely to have full column rank so that the Primal Range-Space Condition is satisfied, under which the least squares problem (1.8) can be transformed to $F(\tilde{z}, v, y)$ (2.6) and $F_y(\tilde{z}, v)$ (2.10).

For a symmetric matrix $x$, let the vector $\lambda(x)$ denote its eigenvalues. The following measures are used to describe the optimality and infeasibility of the iterations:

$$\text{obj} \equiv \|Ax - b\|^2, \quad r_{eq} \equiv \|Ex - f\|_1, \quad r_{ieq} \equiv \|\|Gx - h\|_+\|_1, \quad r_{psd} \equiv \|[-\lambda(x) - d]_+\|_1.$$ 

In the figures below, “eq” is the value of $r_{eq}$, “ieq” is the value of $r_{ieq}$, and “psd” is the value of $r_{psd}$.

Figure 5.1 depicts the progress of Nesterov’s smooth method on reformulation $F(\tilde{z}, v, y)$ (2.6); Figure 5.2 shows the progress of Nesterov’s smooth method on reformulation $F_i(\tilde{z}, v)$ (2.10); Figure 5.3 describes Nesterov’s Excessive Gap METHOD 1 on reformulation $F(\tilde{z}, v, y)$ (2.6); Figure 5.4 illustrates Nesterov’s smooth method on the original model:

$$\min_{x \in \mathbb{S}^n_+} \|Ax - b\|^2 + \max_{y \in \mathbb{R}^m, \nu \geq 0} \langle y, Ex - f \rangle + \langle v, Gx - h \rangle.$$ 

Nesterov’s smooth method and Nesterov’s Excessive Gap METHOD 2 on reformulations $F(\tilde{z}, v, y)$ (2.6) and $F_y(\tilde{z}, v)$ (2.10) are faster than Nesterov’s smooth method [22] on the original formulation (5.1) on this example.
Figure 5.1: Nesterov’s Smooth Method On Dual $F (2.6)$

Figure 5.2: Nesterov’s Smooth Method On Succinct Dual $F_{z,v} (2.10)$
Figure 5.3: Nesterov’s Excessive Gap Method 2 on Dual $F$ (2.6)

Figure 5.4: Original Nesterov’s Smooth Method on Original Dual (5.1)
5.2 H-Norm Nearest Correlation Matrix Problem

This example is the nearest correlation matrix problem in Hadamard product norm. It is built on the E5 class test problems in [30]. For each instance, the operator $A$ and the matrix $b$ are generated in the same way as that in [30]. The number of equalities are the same, except that the coefficient matrix of equality constraints $E$ is dense instead of sparse—if the algorithms work on dense problems, they can be extended to sparse problems using sparsity techniques, and sparsity is not the focus of this paper. Additional inequality constraints are added to the problem instances. The number of inequality constraints is the same as that of equality constraints. And the coefficient matrix of inequality constraints $G$ is dense, too.

To be consistent with the relative infeasibility measures used in [31], the following measures are used in this and the following examples:

$$
obj \equiv \|Ax - b\|^2, \quad r_{eq} \equiv \frac{\|Ex - f\|_1}{\|f\|_1 + 1}, \quad r_{ieq} \equiv \frac{\|[Gx - h]\|_1}{\|h\|_1}, \quad r_{psd} \equiv \frac{\|[\lambda(x - d)]_+\|_1}{\|A\|_1 + 1}.
$$

For this example, modified Nesterov’s Excessive Gap Method 1 is applied to model (2.1) (§§4.2), and standard SDP software are applied to model (1.8) through the YALMIP-20120609 interface.

In the tables below, “it” is the number of iterations; the rows “Gap1-1” contain the results obtained by the modified Nesterov’s Excessive Gap Method 1 using parameter set 1 defined in equations (4.3); the rows “Gap1-2” contain the results generated by the modified Nesterov’s Excessive Gap Method 1 using parameter set 2 defined in equations (4.4).

### Table 1: Nearest Correlation Matrix Problem in Hadamard Product Norm - Ex 1

<table>
<thead>
<tr>
<th>Methods</th>
<th>obj</th>
<th>eq</th>
<th>ieq</th>
<th>psd</th>
<th>CPU (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0 1000 10000</td>
<td>0 1000 10000</td>
<td>0 1000 10000</td>
<td>0 1000 10000</td>
<td>1000 10000</td>
</tr>
<tr>
<td>SDPT3</td>
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<td>1.7e-10</td>
<td>0</td>
<td>0</td>
<td>7888</td>
</tr>
<tr>
<td>Sedumi</td>
<td>crashed</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gap1-1</td>
<td>140.1</td>
<td>1.3e+3</td>
<td>1.2e+3</td>
<td>1</td>
<td>1.1e-5</td>
</tr>
<tr>
<td>Gap1-2</td>
<td>140.0</td>
<td>1.3e+3</td>
<td>1.2e+3</td>
<td>1</td>
<td>1.1e-5</td>
</tr>
</tbody>
</table>

### Table 2: Nearest Correlation Matrix Problem in Hadamard Product Norm - Ex 2

<table>
<thead>
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<th>ieq</th>
<th>psd</th>
<th>CPU (secs)</th>
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</thead>
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<td>0 1000 10000</td>
<td>1000 10000</td>
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<td></td>
<td></td>
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<tr>
<td>Gap1-1</td>
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<td>2.7e+3</td>
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<td>1.6e-5</td>
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<tr>
<td>Gap1-2</td>
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<td>2.7e+3</td>
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### Table 3: Nearest Correlation Matrix Problem in Hadamard Product Norm - Ex 3

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<th>ieq</th>
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<td>1000 10000</td>
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<td>Gap1-2</td>
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<td></td>
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<tr>
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<td>6.0e+3</td>
<td>6.0e+3</td>
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</tr>
<tr>
<td>Gap1-2</td>
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<td>6.0e+3</td>
<td>6.0e+3</td>
<td>1</td>
<td>139.4</td>
</tr>
</tbody>
</table>

In the above examples, the CPU time used by the proposed approach to obtain a relatively accurate solution is only about 1.2 ~ 2.4% of that by SDPT3, while Sedumi crashed on the examples. With the increase in the numbers of equalities and inequalities, the proposed approach is more and more advantageous to second-order methods, because in a second-order method, the dimension of the Newton system increases with the numbers of equalities and inequalities.

### 5.3 Accurate Solutions

In this example, each matrix has 4 sub-matrices with dimensions given in vector \( N \), and \( A \) is a square matrix with dimension \( p = \frac{N_d(N_d+1)}{2} \). In the tables below, the rows “Smooth” contain the results generated by Nesterov’s smooth method on reformulation \( F(\tilde{z},v,y) \) (2.6); the rows “Gap 2” contain the results generated by Nesterov’s Excessive Gap Method 2 on reformulation \( F(\tilde{z},v,y) \) (2.6). By the definition of primal-dual nondegeneracy and strict complementarity for SDP [2, Theorem 12], when the number of constraints is much smaller than the dimension of the variable, the problem instance is likely to be dual degenerate. The approach proposed in this paper gets more accurate solutions than SDPT3 or Sedumi do on the examples because these instances are likely to be dual degenerate.

### Table 5: Random Problem 1

<table>
<thead>
<tr>
<th>Methods</th>
<th>obj</th>
<th>eq</th>
<th>ieq</th>
<th>psd</th>
<th>CPU (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDPT3</td>
<td>1.1e+8</td>
<td>1</td>
<td>4.2-1</td>
<td>0</td>
<td>99.5</td>
</tr>
<tr>
<td>Sedumi</td>
<td>1.1+8</td>
<td>1</td>
<td>4.2e-1</td>
<td>7.8e-11</td>
<td>10.9</td>
</tr>
<tr>
<td>Smooth</td>
<td>5.3e-2</td>
<td>1.4e-5</td>
<td>0</td>
<td>0</td>
<td>28.8</td>
</tr>
<tr>
<td>Gap 2</td>
<td>5.3e-2</td>
<td>1.4e-5</td>
<td>0</td>
<td>0</td>
<td>28.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Methods</th>
<th>obj</th>
<th>eq</th>
<th>ieq</th>
<th>psd</th>
<th>CPU (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDPT3</td>
<td>1.7e+10</td>
<td>1</td>
<td>4.8-1</td>
<td>0</td>
<td>7091.5</td>
</tr>
<tr>
<td>Sedumi</td>
<td>2.6e+21</td>
<td>2.6e+6</td>
<td>5.4e+5</td>
<td>1.2e+6</td>
<td>124.9</td>
</tr>
<tr>
<td>Smooth</td>
<td>4.4e-2</td>
<td>1.9e-4</td>
<td>0</td>
<td>0</td>
<td>247.1</td>
</tr>
<tr>
<td>Gap 2</td>
<td>4.4e-2</td>
<td>1.9e-4</td>
<td>0</td>
<td>0</td>
<td>250.9</td>
</tr>
</tbody>
</table>

### Table 6: Random Problem 2

<table>
<thead>
<tr>
<th>Methods</th>
<th>obj</th>
<th>eq</th>
<th>ieq</th>
<th>psd</th>
<th>CPU (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDPT3</td>
<td>1.1e+8</td>
<td>1</td>
<td>4.2-1</td>
<td>0</td>
<td>99.5</td>
</tr>
<tr>
<td>Sedumi</td>
<td>1.1+8</td>
<td>1</td>
<td>4.2e-1</td>
<td>7.8e-11</td>
<td>10.9</td>
</tr>
<tr>
<td>Smooth</td>
<td>5.3e-2</td>
<td>1.4e-5</td>
<td>0</td>
<td>0</td>
<td>28.8</td>
</tr>
<tr>
<td>Gap 2</td>
<td>5.3e-2</td>
<td>1.4e-5</td>
<td>0</td>
<td>0</td>
<td>28.2</td>
</tr>
</tbody>
</table>

In the above examples, the CPU time used by the proposed approach to obtain a relatively accurate solution is only about 1.2 ~ 2.4% of that by SDPT3, while Sedumi crashed on the examples. With the increase in the numbers of equalities and inequalities, the proposed approach is more and more advantageous to second-order methods, because in a second-order method, the dimension of the Newton system increases with the numbers of equalities and inequalities.

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### Table 7: Random Problem 2

<table>
<thead>
<tr>
<th>Methods</th>
<th>obj</th>
<th>eq</th>
<th>ieq</th>
<th>psd</th>
<th>CPU (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDPT 3</td>
<td>3.8e+11</td>
<td>1</td>
<td>4.9-1</td>
<td>0</td>
<td>25786</td>
</tr>
<tr>
<td>Sedumi</td>
<td>crashed</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Smooth</td>
<td>2.8e-3</td>
<td>2.5e-4</td>
<td>0</td>
<td>0</td>
<td>993.5</td>
</tr>
<tr>
<td>Gap 2</td>
<td>7.3e-6</td>
<td>2.9e-2</td>
<td>5.1e-3</td>
<td>1.6e-5</td>
<td>961.9</td>
</tr>
</tbody>
</table>

### Table 8: Random Problem 2

<table>
<thead>
<tr>
<th>Methods</th>
<th>obj</th>
<th>eq</th>
<th>ieq</th>
<th>psd</th>
<th>CPU (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDPT 3</td>
<td>NA</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sedumi</td>
<td>crashed</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Smooth</td>
<td>2.0e-2</td>
<td>1.4e-5</td>
<td>0</td>
<td>0</td>
<td>4571.9</td>
</tr>
<tr>
<td>Gap 2</td>
<td>2.0e-2</td>
<td>1.4e-5</td>
<td>0</td>
<td>0</td>
<td>4402.3</td>
</tr>
</tbody>
</table>

### Acknowledgements

This research was supported in part by Université Joseph Fourier, Pôle Math-STIC. I would like to thank the host Jérôme Malick for hospitality and discussions. I also would like to thank Professor Yurii Nesterov for suggestions that improve the paper. As well, I would like to thank the editor and anonymous reviewers for very helpful comments.

### A Moore-Penrose Generalized Inverse

For a possibly rectangular matrix $M$, let $M^\dagger$ denote its Moore-Penrose generalized inverse, which is a unique solution to the following four equations:

(A.1) \[ MXM = M, \quad XMX = X \]

(A.2) \[ (MX)^* = MX, \quad (XM)^* = XM. \]

If $M$ is square and non-singular, we have $M^\dagger = M^{-1}$.

Below are some properties of Moore-Penrose generalized inverse [25] used in this paper:

(A.3) \[ M^\dagger M^\dagger M^\dagger M = M^\dagger = M^* M^\dagger M^\dagger, \quad M^\dagger M^\dagger M^\dagger M^\dagger M = M^* = M^* M M^\dagger \]

(A.4) \[ M^\dagger = M, \quad M^\dagger = M^*, \quad (M^* M)^\dagger = M^\dagger M^\dagger \]

(A.5) \[ M^\dagger, M M^\dagger, I - M^\dagger M, I - MM^\dagger \] are all hermitian idempotent.

In addition, let the singular value decomposition of $M$ be

\[ M = U \Sigma V^*, \]

where $U$ and $V$ are unitary and $\Sigma$ is rectangular diagonal of the same size as $M$ with non-negative real diagonal entries. Then

\[ M^\dagger = V \Sigma^\dagger U^*, \]

where $\Sigma^\dagger$ is obtained from $\Sigma$ by replacing each positive diagonal entry by its reciprocal.

Let $P_M$ be the projection onto the range space of $M$. If $M \in \mathbb{C}^{n \times m}$ is of rank $m$, we have

(A.6) \[ M^\dagger = (M^* M)^{-1} M^*, \quad M^\dagger M = I, \quad MM^\dagger = P_M. \]

References


